

A STRUCTURAL CONJECTURE FOR THE INFINITUDE OF TWIN PRIMES

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ABSTRACT. We develop a structural reformulation of the Twin Prime Conjecture (TPC) in terms of additive closure properties of the set of twin prime witnesses. Every prime above 3 lies in one of two arithmetic channels; a *witness* w is a positive integer for which both $6w - 1$ and $6w + 1$ are prime. Let \mathcal{W} denote the set of all witnesses. The central result is

$$\text{BC} \implies \text{TPC},$$

proved unconditionally. Here BC (Bridging Conjecture) asserts that for every threshold $V \geq 1$, some pair $u, v \in \mathcal{W}$ with $u, v \leq V$ sums to a witness $w > V$. Under the Witness Decomposition Conjecture (WDC), $\text{BC} \Leftrightarrow \text{TPC}$. No analytic hypotheses appear anywhere in the paper.

The structural results stand independently of any analytic or elementary approach to BC; companion papers explore such routes separately.

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PRELUDE: DUBNER'S BALL

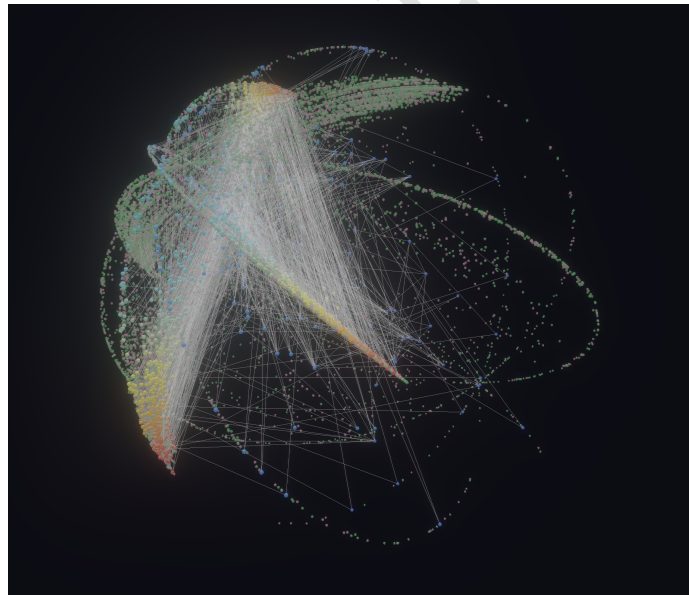


FIGURE 1. The Dubner Ball at $|\mathcal{W}| = 20,000$ witnesses. Each point is a twin prime witness, coloured by residue class modulo 5 ($k = 0$: yellow/green; $k = 2$: blue/cyan; $k = 3$: red/orange). Witnesses with $k \in \{1, 4\}$ are absent by a theorem provable from $6 \equiv 1 \pmod{5}$ alone. Filaments connect each witness to its progenitor pairs — the pairs $(u, v) \in \mathcal{W} \times \mathcal{W}$ with $u + v = w$. The structure is not random.

This paper is dedicated to the memory of Harvey Dubner (1928–2019), whose Middle Number Conjecture is the load-bearing hypothesis of this work.

Dubner spent decades building custom hardware to search for large primes and prime constellations. Among his many contributions, he conjectured that the set \mathcal{W} of twin prime witnesses — positive integers w such that $6w - 1$ and $6w + 1$ are both prime — is *additively self-generating*: that every witness $w > 1$ can be written as $u + v$ for some $u, v \in \mathcal{W}$. This conjecture, sometimes called the Middle Number Conjecture, is the load-bearing hypothesis of this paper. Dubner did not live to see its fate resolved. The present work is one attempt to honour his intuition by developing the algebraic infrastructure that makes its consequences precise.

The object in Figure 1 — which we call the *Dubner Ball* [3] — is the directed graph whose nodes are the elements of \mathcal{W} and whose edges connect each witness to its progenitor pairs. It lives naturally in three dimensions, organised by the three active residue classes of \mathcal{W} modulo 5. The ball is not a heuristic: the absence of witnesses with $k \equiv 1, 4 \pmod{5}$ is a theorem, proved from $6 \equiv 1 \pmod{5}$ alone. The filament structure — the web of additive progenitors — is the visible signature of what Dubner conjectured.

The ball suggests something deeper. If witnesses are additively self-generating, and if the additive structure of \mathcal{W} reaches across every threshold — bridging each frontier with a new witness built from old ones — then the infinitude of twin primes follows without ever directly asking whether a given large integer is prime. The logical landscape of this idea is developed in [4]. The present paper takes the next step: proving that the Bridging Conjecture alone is sufficient to imply TPC, and isolating BC as the unique remaining open gap.

A NOTE ON THIS EDITION

This is the **academic edition**, prepared for journal submission. It contains the mathematics only. The Lean 4 machine-verified proofs and the literary elements of the source edition have been suppressed.

The **source edition** — which includes the full document with Lean 4 proofs, literary voice, and the Interlude — and the **literate edition** — which pairs the mathematics with the machine-verified proofs — are published at wildducktheories.github.io/twin-primes.

A note on section numbering. The section numbers in this paper are not consecutive. This is intentional. The sections are numbered according to OEIS sequence A002822 — the twin prime witnesses, the

very objects of study — whose terms are 1, 2, 3, 5, 7, 10, 12, 15, 18, 20, ... The gaps in the numbering (4, 6, 8, 9, 11, ...) are occupied by the complementary sequence A067611, which indexes the *source edition's* literary and allegorical sections.

Together the two sequences cover all of \mathbb{N} . The chapter numbering of this paper *is* the mathematics: A002822 and A067611 interleaved over the natural numbers, as they are throughout the theory.

We draw attention to the interleaved nature of A002822 and A067611 and, in the interests of maintaining the formal mathematical character of this edition, we refer the interested reader to a fuller explanation in the source edition.

1. INTRODUCTION

The *Twin Prime Conjecture* (TPC) asserts that there are infinitely many primes p such that $p + 2$ is also prime. Every such pair has the form $(6w - 1, 6w + 1)$ for a unique positive integer w , called a *twin prime witness*. Let $\mathcal{W} \subset \mathbb{N}$ denote the set of all witnesses (OEIS A002822). TPC is equivalent to the assertion that \mathcal{W} is infinite.

This paper develops a structural reformulation of TPC in which the remaining obstruction is isolated into a single precise additive closure conjecture, BC, with no analytic hypotheses anywhere in the chain. The central result is:

$$\text{BC} \implies \text{TPC}.$$

This implication is proved unconditionally. $\text{BC} \implies \text{TPC}$ is Theorem 10.6; under WDC, $\text{BC} \Leftrightarrow \text{TPC}$ is Theorem 10.7. BC itself remains open. An elementary route toward BC via Bertrand's Postulate is developed in a companion paper [17].

The paper carefully separates unconditional algebraic identities from conjectural structural closure principles. No sieve-theoretic density estimates are used in the derivation of the structural implications themselves. Classical analytic approaches (sieve theory [5], GPY/Maynard [7, 8]) work directly on primes and encounter the parity obstruction. The structural approach reframes the problem: rather than detecting primality via sieve cancellation, it asks whether the *additive closure* of \mathcal{W} is sufficient to guarantee its infinitude.

Line 1 — Additive structure (purely algebraic, no analytic input): the Witness Decomposition Conjecture (WDC) and the Bridging Conjecture (BC) assert additive closure properties of \mathcal{W} at global and threshold scales respectively. Their proved implications yield $\text{BC} \implies \text{TPC}$ unconditionally.

Line 2 — Channel exhaustion (bridge to the analytic world): a diagnostic tool showing precisely where the structural approach hands off to analysis. Channel exhaustion sits nearer to classical analytic questions than WDC or BC, but connects the additive structure of \mathcal{W} to the broader distribution of primes via the channel representation.

The Gap Conjecture (GC) — that consecutive witnesses satisfy $w_{n+1} \leq 2w_n$ — is a metric consequence of the structural conjectures, not a foundational assumption. Both WDC and BC independently imply GC (Section 7); GC implies neither. It is named and stated for reference but is not a load-bearing conjecture in the argument.

The paper is organised as follows. Section 2 introduces the two-channel representation and the defect forms. Section 3 states and proves the Multiplexing Identity. Section 5 establishes algebraic constraints on witnesses via forbidden residues and arithmetic progression bounds. Section 7 presents GC as a metric corollary. Section 10 develops Line 1: the additive structure of witnesses, WDC, BC, and their proved implications. Section 20 develops Line 2: channel exhaustion as the analytic bridge. Section 22 assembles the full implication chain and identifies BC as the unique remaining gap. Sections 15 and 18 develop the Bridging Machine and Gap Machine, making the structural dynamics explicit. Section 27 situates the paper in the literature. Section 28 concludes. The appendix provides a logical dependency map of all results.

Part 1. Foundations

2. THE CHANNEL REPRESENTATION

Definition 2.1. For $m \geq 1$, define the *lower channel value* $L_m := 6m - 1$ and the *upper channel value* $H_m := 6m + 1$.

Remark 2.2 (Why channels?). The integers $\equiv -1 \pmod{6}$ and $\equiv +1 \pmod{6}$ form two parallel arithmetic progressions — two infinite *channels* through which all primes above 3 must flow. The index m parameterises both channels simultaneously: L_m and H_m are the unique elements at position m in the lower and upper channel respectively, and they always differ by exactly 2. A twin prime pair is precisely a position m at which both channels simultaneously carry a prime.

The term *multiplexing* follows naturally: adding two witnesses u and v combines their channel signals into a single new index $u + v$, with the Multiplexing Identity (Section 3) describing exactly how the channel values combine.

Every prime $p > 3$ satisfies $p \equiv \pm 1 \pmod{6}$, so every prime $p > 3$ is either a lower channel value L_m or an upper channel value H_m for a unique $m \geq 1$. The channels are *universal for primes*: they cover every prime above 3.

Lemma 2.3 (Primes in channels). *Every prime $p > 3$ lies in a channel: there exists $m \geq 1$ such that $p = L_m$ or $p = H_m$.*

Proof. Since $p > 3$ is prime, $\gcd(p, 6) = 1$, so $p \equiv 1$ or $p \equiv 5 \pmod{6}$. If $p \equiv 1 \pmod{6}$: set $m = (p - 1)/6 \geq 1$; then $H_m = 6m + 1 = p$. If $p \equiv 5 \pmod{6}$: set $m = (p + 1)/6 \geq 1$; then $L_m = 6m - 1 = p$. \square

Remark 2.4. Lemma 2.3 is formally verified in Lean 4 (`prime_in_channel`, zero sorry).

Twin prime pairs $(p, p+2)$ with $p > 3$ are precisely the pairs (L_m, H_m) for some $m \geq 1$: the two channels satisfy $H_m - L_m = 2$ for all m . The channels are universal, but witnesses isolate those channel indices m supporting *simultaneous* primality in both channels.

Definition 2.5. A positive integer m is a *witness* if L_m and H_m are both prime. Let \mathcal{W} denote the set of all witnesses. The base witness is $\omega := 1$ ($L_1 = 5$, $H_1 = 7$). A positive integer $m \notin \mathcal{W}$ is a *defect*; let $\mathcal{D} := \mathbb{N}_{>0} \setminus \mathcal{W}$. Let \mathbb{P} denote the set of all prime numbers.

Remark 2.6. The Twin Prime Conjecture is equivalent to the assertion that \mathcal{W} is infinite.

2.1. The defect forms. Since every prime factor of L_m or H_m is itself $\equiv \pm 1 \pmod{6}$, composite channel values factorise in a constrained way. The mod-6 product table,

$$(6a-1)(6b-1) \equiv +1, \quad (6a+1)(6b+1) \equiv +1, \quad (6a-1)(6b+1) \equiv -1 \pmod{6},$$

yields the following complete characterisation of defects.

Proposition 2.7 (Defect forms). *Let $m \geq 1$.*

- (1) *If L_m is composite, then $m = 6ab + a - b$ for some $a, b \geq 1$. (Type $-+$)*
- (2) *If H_m is composite with both prime factors $\equiv -1 \pmod{6}$, then $m = 6ab - a - b$ for some $a, b \geq 1$. (Type $--$)*
- (3) *If H_m is composite with both prime factors $\equiv +1 \pmod{6}$, then $m = 6ab + a + b$ for some $a, b \geq 1$. (Type $++$)*

Consequently, m is a witness if and only if it belongs to none of these three forms for any $a, b \geq 1$.

Proof. Type \dashv : $(6a - 1)(6b + 1) = 36ab + 6a - 6b - 1$, so $6m - 1 = 36ab + 6a - 6b - 1$ gives $m = 6ab + a - b$. The remaining types follow by expanding $(6a - 1)(6b - 1)$ and $(6a + 1)(6b + 1)$. \square

Remark 2.8. Proposition 2.7 was independently discovered by Balestrieri [2], who states it as Lemma 2.1 and uses it to reformulate TPC as: *there exist infinitely many n simultaneously avoiding all three defect forms*. The present paper develops this characterisation further via the Multiplexing Identity (Theorem 3.1).

Non-witnesses are not random: they fall into exactly three algebraic families. This is the first structural result of the paper and the foundation on which the Multiplexing Identity, the additive conjectures, and the channel exhaustion argument all rest.

3. THE MULTIPLEXING IDENTITY

The Multiplexing Identity is the central algebraic mechanism of the paper. It expresses the channel values of a sum $w = u + v$ directly in terms of those of u and v , enabling every later result to be stated in terms of additive structure on \mathcal{W} .

Theorem 3.1 (Multiplexing Identity). *For all $u, v \geq 1$:*

$$\begin{aligned} (1) \quad & L_{u+v} = L_u + L_v + 1, \\ (2) \quad & H_{u+v} = H_u + H_v - 1. \end{aligned}$$

Proof. Direct computation:

$$L_u + L_v + 1 = (6u - 1) + (6v - 1) + 1 = 6(u + v) - 1 = L_{u+v}.$$

Similarly $H_u + H_v - 1 = (6u + 1) + (6v + 1) - 1 = 6(u + v) + 1 = H_{u+v}$. \square

3.1. Cross-term collapse and channel decoupling.

Corollary 3.2 (Cross-term collapse). *For all $u, v \geq 1$:*

$$L_u + H_v = H_u + L_v = 6(u + v).$$

In particular, both cross pairings are divisible by 6 and hence composite for $u + v > 1$.

Proof. $(6u - 1) + (6v + 1) = 6(u + v)$. Likewise for the other pairing. \square

Remark 3.3. Cross-term collapse induces a structural decoupling: mixed-channel sums are forced composite, leaving only like-channel additive interactions as potential witness generators. The four pairings of (L_u, H_u) with (L_v, H_v) split into two *parallel* outputs (L_{u+v}, H_{u+v}) and two *cross* outputs $(6(u+v), 6(u+v))$. The cross outputs are always absorbed by 6; only the parallel channels carry primality information.

3.2. The shift form.

Corollary 3.4 (Shift form). *For all $u, v \geq 1$:*

$$L_{u+v} = L_u + 6v = 6u + L_v, \quad H_{u+v} = H_u + 6v = 6u + H_v.$$

Proof. $L_u + 6v = (6u - 1) + 6v = 6(u + v) - 1 = L_{u+v}$. The upper channel follows analogously. \square

Remark 3.5. Adding v to u shifts the channel pair (L_u, H_u) rigidly upward by $6v$. The twin-prime gap $H_{u+v} - L_{u+v} = 2$ is preserved under addition.

3.3. The witness condition via the Multiplexing Identity.

Corollary 3.6 (Witness condition). *Let $w = u + v$ with $u, v \geq 1$. Then $w \in \mathcal{W}$ if and only if*

$$L_u + L_v + 1 \in \mathbb{P} \quad \text{and} \quad H_u + H_v - 1 \in \mathbb{P}.$$

Remark 3.7. The Multiplexing Identity translates the question “is w a witness?” into a Diophantine constraint on the channel values of its additive decompositions. This is the foundation of the additive structural approach: the primality question is never asked directly about w , but about sums of channel values of smaller integers.

4. A NOTE ON SECTION NUMBERING

The section numbers of this paper follow OEIS A002822 (twin prime witnesses) and A067611 (their complement). The rationale — and the off-by-one artefact introduced by the well-known difficulties of implementing chromosomal numbering schemes in \LaTeX — are explained in the source edition, which is not missing a vital chromosome.

5. ALGEBRAIC CONSTRAINTS ON WITNESSES

The defect forms and the Multiplexing Identity determine not just the structure of individual witnesses but the global constraints that \mathcal{W} must satisfy as an arithmetic set. This section establishes those constraints unconditionally. They constitute local obstructions on witness configurations and provide the first evidence that \mathcal{W} behaves as a highly structured arithmetic object rather than a random sparse subset of \mathbb{N} .

5.1. Forbidden residues.

Lemma 5.1 (Forbidden residues). *For a prime $p \geq 5$, define the two forbidden residues:*

$$r_L(p) := 6^{-1} \pmod{p}, \quad r_H(p) := -6^{-1} \pmod{p}.$$

Then:

$$p \mid L_m \iff m \equiv r_L(p) \pmod{p}, \quad p \mid H_m \iff m \equiv r_H(p) \pmod{p}.$$

The two forbidden residues are distinct for all $p \geq 5$: they differ by $-2 \cdot 6^{-1} \pmod{p}$, which is nonzero.

Proof. $p \mid 6m - 1 \iff 6m \equiv 1 \pmod{p} \iff m \equiv 6^{-1} \pmod{p}$. Similarly $p \mid 6m + 1 \iff m \equiv -6^{-1} \pmod{p}$. The difference $6^{-1} - (-6^{-1}) = 2 \cdot 6^{-1}$; this is nonzero mod p since $p \geq 5$ implies $\gcd(12, p) = 1$. \square

5.2. The arithmetic progression bound.

Theorem 5.2 (AP length bound). *Let $a, a+d, a+2d, \dots, a+(k-1)d$ be a maximal arithmetic progression in \mathcal{W} with common difference $d \geq 1$. Let p be any prime with $p \nmid d$. Then $k < p$.*

Proof. Since $p \nmid d$, the residues $a, a+d, a+2d, \dots \pmod{p}$ cycle through all p residue classes as the index increases. Within any p consecutive terms, one term has residue $r_L(p)$ and another has residue $r_H(p)$. By Lemma 5.1, both terms have a composite channel value and are therefore defects. Hence no p consecutive terms can all be witnesses; $k < p$. \square

Corollary 5.3. *An AP in \mathcal{W} of length k requires that every prime $p < k$ divides d . Equivalently, $\text{lcm}(2, 3, \dots, k-1) \mid d$.*

Remark 5.4. These constraints are unconditional and finitely checkable. Theorem 5.2 is tight: empirically, APs of length exactly $p-1$ exist for small primes p , achieved when d is divisible by all primes smaller than p .

5.3. The mod-5 structure. The smallest prime ≥ 5 is 5 itself. Since $6 \equiv 1 \pmod{5}$, Lemma 5.1 takes a particularly clean form.

Theorem 5.5 (Mod-5 structure). *For $m > 1$:*

- (1) $m \equiv 1 \pmod{5} \implies L_m \equiv 0 \pmod{5}$, so $m \in \mathcal{D}$.
- (2) $m \equiv 4 \pmod{5} \implies H_m \equiv 0 \pmod{5}$, so $m \in \mathcal{D}$.

Consequently, $\mathcal{W} \setminus \{\omega\} \subseteq \{m \in \mathbb{N} : m \not\equiv 1, 4 \pmod{5}\}$.

Proof. $6 \equiv 1 \pmod{5}$, so $L_m = 6m - 1 \equiv m - 1 \pmod{5}$, which vanishes iff $m \equiv 1 \pmod{5}$. For $m > 1$, $L_m > 5$, so $m \in \mathcal{D}$. Similarly, $H_m \equiv m + 1 \pmod{5}$, vanishing iff $m \equiv 4 \pmod{5}$. \square

Corollary 5.6 (Mod-5 partition closure). *Let $w = u + v$ with $u, v \in \mathcal{W} \setminus \{\omega\}$. Then $w \in \mathcal{W}$ only if $(u \pmod{5}, v \pmod{5}) \notin \{(2, 2), (3, 3)\}$. Among the six unordered pairs from the active residue classes $\{0, 2, 3\}$ of $\mathcal{W} \pmod{5}$, exactly two are forbidden:*

$u \pmod{5}$	$v \pmod{5}$	$(u + v) \pmod{5}$	
0	0	0	✓
0	2	2	✓
0	3	3	✓
2	3	0	✓
2	2	4	×
3	3	1	×

Proof. The active residue classes of $\mathcal{W} \setminus \{\omega\}$ modulo 5 are $\{0, 2, 3\}$ by Theorem 5.5. Among the six unordered pairs, $(2, 2)$ gives $w \equiv 4$ and $(3, 3)$ gives $w \equiv 1$ — both excluded from $\mathcal{W} \setminus \{\omega\}$. \square

Remark 5.7. Corollary 5.6 gives the first unconditional constraint on which partitions $w = u + v$ can be witness partitions. The two forbidden types are not merely absent in practice — they are provably impossible, following from $6 \equiv 1 \pmod{5}$ alone. Higher sieving primes ($p = 7, 11, \dots$) impose further constraints of the same character.

Remark 5.8 (Distribution within the support). Theorem 5.5 determines the *support* of $\mathcal{W} \pmod{5}$: the active residue classes are $\{0, 2, 3\}$. A natural question is whether the *distribution within the support* is uniform, or whether a Chebyshev-style bias is present — specifically, whether class 3 persistently outnumbers class 2.

This conjecture was posed on structural grounds: the base witness $\omega = 1$ lies in class 1 $\pmod{5}$ and has access to sums $1 + w \equiv 3 \pmod{5}$ (for $w \equiv 2$) that no element of $\{0, 2, 3\}$ can produce, suggesting an asymmetric pressure toward class 3. The conjecture is conditionally denied by the empirical evidence of Section 25 (Observation 25.2): the distribution over $\{0, 2, 3\} \pmod{5}$ is statistically uniform to $N = 10^6$ ($\chi^2 = 0.67$, $p = 0.72$). The structural premise for the conjecture is real; the predicted consequence is not observed.

5.4. Structural sparsity of witnesses. The constraints of this section are individually elementary, but their cumulative effect is significant: they force witnesses to be *structurally sparse*, not merely heuristically rare.

Remark 5.9 (Defect density). The witness set \mathcal{W} has natural density 0: a witness w requires both $6w-1$ and $6w+1$ to be prime simultaneously, and the count of such $w \leq M$ is $O(M/\log^2 M)$ [5]. This classical fact is used below but plays no role in the main implication chain.

Corollary 5.10 (Large gaps are structurally generic). *For any K , the proportion of witnesses $w \leq M$ for which the next witness satisfies $w' - w \leq K$ tends to zero as $M \rightarrow \infty$. Equivalently, for almost all witnesses w (in the sense of natural density), the gap to the next witness exceeds any fixed bound K .*

Proof. Witnesses have density 0 by Remark 5.9. A set of density 0 cannot have a positive-density subset with bounded gaps: bounded gaps would force density $\geq 1/K > 0$. \square

Remark 5.11. Corollary 5.10 is an unconditional structural statement: it does not assume TPC, WDC, BC, or any conjecture. It says that *large gaps between witnesses are the norm, not the exception*, purely from the defect form characterisation. This is the precise sense in which BC, if proved, would be non-trivial: it asserts that despite generic large gaps, witnesses *always* manage to bridge every threshold — a global closure property that the local sparsity makes no promise of.

Note that Corollary 5.10 concerns the *almost-all* behaviour of gaps. It does not rule out the Gap Conjecture ($w_{n+1} \leq 2w_n$), which concerns the *worst-case* gap. The two statements are compatible: gaps can grow without bound on average while still remaining below $2w_n$ in every individual case.

The observation admits two precise formulations as conjectures.

Conjecture 5.13 (Asymmetric decomposition). *For every $w \in \mathcal{W}$ with $w > \omega$, let $v(w)$ denote the largest witness $v < w$ such that $w - v \in \mathcal{W}$. Then*

$$\frac{w - v(w)}{w} \rightarrow 0 \quad \text{as } w \rightarrow \infty.$$

That is, the gap between a witness and its largest valid summand is $o(w)$: the dominant decomposition places the larger summand close to w , not near $w/2$.

Remark 5.14. Under WDC, $w = u + v$ with $u \leq v$, so $v \geq w/2$ always. GC gives $w_{n+1} \leq 2w_n$, i.e. the gap $w_{n+1} - w_n \leq w_n$ in the worst case. Conjecture 5.13 asserts something stronger: the gap $w - v(w)$ between a witness and its largest valid summand is not merely $\leq w/2$ but $o(w)$. In other words, valid decompositions are not uniformly distributed between $w/2$ and w — the defect constraints push $v(w)$ close to w , leaving a gap $w - v(w)$ that is small relative to w .

This is a direct structural consequence of the defect pressure for separation: most values near $w/2$ are defects, so the largest valid v is pushed toward w rather than pulled back toward $w/2$. Conjecture 5.13 presupposes WDC and strengthens it. It is not a consequence of GC; rather, it predicts that the actual gap $w_{n+1} - w_n$ is $\ll w_n$ for almost all n , consistent with but strictly stronger than GC’s worst-case bound.

Conjecture 5.15 (Unbounded average gap). *The average gap between consecutive witnesses diverges:*

$$\frac{1}{|\mathcal{W} \cap [1, M]|} \sum_{\substack{w_n, w_{n+1} \in \mathcal{W} \\ w_{n+1} \leq M}} (w_{n+1} - w_n) \rightarrow \infty \quad \text{as } M \rightarrow \infty.$$

Remark 5.16. Conjecture 5.15 strengthens Corollary 5.10 from “gaps exceed any fixed bound for almost all witnesses” to “the average gap itself grows without bound.” It is consistent with GC ($w_{n+1} \leq 2w_n$): the worst-case gap can be bounded by w_n while the average still diverges if witnesses grow fast enough. Conjecture 5.15 follows immediately from the Hardy–Littlewood conjecture on twin prime density ($|\mathcal{W} \cap [1, M]| \sim CM/\log^2 M$), but can be stated purely in terms of the structure of \mathcal{W} without appealing to analytic number theory.

7. METRIC CONSEQUENCES OF STRUCTURAL CLOSURE

Before introducing the structural conjectures, we identify what the algebraic infrastructure already implies about the *spacing* of witnesses — independently of any conjecture.

Conjecture 7.1 (Gap Conjecture, GC). *For all consecutive witnesses $w_n, w_{n+1} \in \mathcal{W}$,*

$$w_{n+1} \leq 2w_n.$$

Remark 7.2. GC is a metric statement about the spacing of witnesses. The analogue for primes — Bertrand’s postulate, $p_{n+1} < 2p_n$ — is a theorem; the witness analogue is open. GC says nothing about how witnesses are *constructed* from other witnesses; that is the domain of the structural conjectures in Section 10.

The following results show that GC is a *consequence* of either structural conjecture, not an independent assumption. This separation is deliberate: GC is treated as a metric shadow of structural closure rather than as an independent heuristic principle from which structure is inferred.

Theorem 7.3 (WDC \Rightarrow GC). *The Witness Decomposition Conjecture implies the Gap Conjecture.*

Proof. Fix consecutive witnesses $w_n < w_{n+1}$. By WDC, write $w_{n+1} = u + v$ with $u, v \in \mathcal{W}$ and $u \leq v < w_{n+1}$, so $u \leq v \leq w_n$. Hence $w_{n+1} = u + v \leq 2w_n$. \square

Theorem 7.4 (BC \Rightarrow GC). *The Bridging Conjecture implies the Gap Conjecture (unconditionally).*

Proof. Suppose GC fails: $w_{n+1} > 2w_n$ for some consecutive pair. Set $V = w_n$. BC gives $u, v \in \mathcal{W}$ with $u \leq v \leq w_n$ and $w = u + v \in \mathcal{W}$ with $w > w_n$. But $w \leq 2w_n < w_{n+1}$, placing $w \in (w_n, w_{n+1})$ — contradicting w_{n+1} being the next witness. Hence GC holds. \square

Remark 7.5. Both Theorem 7.3 and Theorem 7.4 are unconditional: GC follows from either structural conjecture without additional hypotheses. The converses — GC \Rightarrow WDC and GC \Rightarrow BC — are both open. Spacing does not imply structure. GC is therefore a named corollary, not a peer conjecture.

Remark 7.6. If GC \Rightarrow WDC were proved, then GC, WDC, and BC would all be equivalent (under TPC), and GC — a single clean inequality on consecutive witnesses — would be the unique obstruction to TPC. This is Conjecture 10.9 of Section 10. It has been verified computationally for all witnesses up to 100,000.

Part 2. Line 1: The Additive Structure of Witnesses

10. LINE 1: THE ADDITIVE STRUCTURE OF WITNESSES

This section is the heart of the paper. The algebraic constraints of Sections 5 and 7 impose local obstructions on witness configurations. The two structural conjectures of this section propose *global* additive closure: that witnesses are, in a precise sense, generative — every witness is a sum of earlier witnesses (WDC), and witnesses always bridge every threshold (BC). These are structural claims, not density claims.

10.1. The structural conjectures.

Conjecture 10.1 (Witness Decomposition Conjecture, WDC). *For every $w \in \mathcal{W}$ with $w > \omega$, there exist $u, v \in \mathcal{W}$ with $u \leq v$ and $u + v = w$.*

Remark 10.2. WDC asserts that \mathcal{W} is *additively self-generating*: every witness beyond the base arises as a sum of two witnesses. Not every sum of two witnesses is a witness; the content is that every witness has at least one such representation. WDC is a restatement of Harvey Dubner’s *Middle Number Conjecture* [1] in the language of witnesses.

Conjecture 10.3 (Bridging Conjecture, BC). *For every $V \in \mathbb{N}$ with $V \geq 1$ there exist $u, v, w \in \mathcal{W}$ with $u \leq v \leq V < w$ and $u + v = w$.*

Remark 10.4. BC asserts that for any threshold V , some pair of witnesses $u, v \leq V$ sums to a witness w strictly above V . The pair (u, v) is a *bridging pair* for V : witnesses already known to lie below the frontier certify the existence of a new witness beyond it.

WDC and BC are siblings: both assert additive closure properties of \mathcal{W} , at different scales. WDC is global (every witness decomposes); BC is threshold-local (decompositions reach across every frontier). WDC is the stronger claim: under TPC, WDC implies BC (Theorem 10.7). BC is closely related to TPC itself: $BC \Rightarrow TPC$ is unconditional (Theorem 10.6).

Remark 10.5 (The strength of the universal quantifier). Both WDC and BC are strict universal statements. WDC does not assert that *almost all* witnesses decompose, or that decomposition holds for sufficiently large w : it asserts that *every* witness beyond the base is a sum of two witnesses, with no exceptions. BC does not assert that bridging pairs exist for a density-one set of thresholds: it asserts that for *every* $V \geq 1$, some pair bridges it. This universality is precisely what makes BC strong enough to imply TPC unconditionally (Theorem 10.6): a single threshold with no bridging pair would terminate the witness sequence.

This is a demanding ask. A weaker density-zero variant (BCZ) also implies TPC, and is discussed in earlier work [4]. BCZ asserts only that bridging fails for at most a density-zero set of thresholds — a statement compatible with finitely many (or even infinitely many sparse) exceptions. BCZ is not pursued further here, for a specific reason: proving BCZ would require an analytic argument controlling the distribution of exceptions, and any such argument faces the same parity obstruction that has blocked sieve methods from the outset. The structural approach gains its leverage precisely from the hard universal form of BC: the Bridging Machine either runs forever or it halts, and WDC either holds for every witness or it does not. These are structural questions, not density questions, and they interact with the defect form analysis in a way that BCZ does not. WDC and BC as stated here make hard, exception-free claims about the additive structure of \mathcal{W} at every scale. Their plausibility rests on the structural evidence of Sections 5 and 15: the defect form analysis constrains which sums can be witnesses, yet has always left enough compatible pairs to bridge every threshold examined.

10.2. The proved implications.

Theorem 10.6 ($BC \Rightarrow TPC$). *The Bridging Conjecture implies the Twin Prime Conjecture.*

Proof. For any $V \in \mathbb{N}$, BC gives $w \in \mathcal{W}$ with $w > V$. Since V is arbitrary, \mathcal{W} is unbounded, hence infinite. \square

Theorem 10.7 ($WDC + TPC \Rightarrow BC$). *The Witness Decomposition Conjecture and the Twin Prime Conjecture together imply the Bridging Conjecture.*

Proof. Assume WDC and TPC (so \mathcal{W} is infinite). Fix any $V \in \mathbb{N}$. Since \mathcal{W} is infinite, there exists a smallest witness $w^* > V$. By WDC, $w^* = u + v$ with $u \leq v$ and $u, v \in \mathcal{W}$. Since $u, v < w^*$ and w^* is the smallest witness above V , we have $u \leq v \leq V$. So (u, v, w^*) is a bridging triple for V . \square

Remark 10.8. Under WDC, $BC \Leftrightarrow TPC$: the forward direction is Theorem 10.6; the reverse is Theorem 10.7. TPC is needed in Theorem 10.7: a finite \mathcal{W} can satisfy WDC (every witness decomposes into earlier ones) yet fail BC at $V = \max \mathcal{W}$. The structural conjecture WDC does not, by itself, force the infinitude of \mathcal{W} .

10.3. The honest asymmetry.

Implication	Status	Notes
$BC \Rightarrow TPC$	Proved	Unconditional
$WDC + TPC \Rightarrow BC$	Proved	Under WDC: $BC \Leftrightarrow TPC$
$WDC \Rightarrow GC$	Proved	Unconditional (Thm 7.3)
$BC \Rightarrow GC$	Proved	Unconditional (Thm 7.4)
$GC \Rightarrow WDC$	Open	Verified to 10^5
$GC \Rightarrow BC$	Open	Spacing does not imply structure
WDC	Conjecture	Global additive closure
BC	Conjecture	Threshold additive closure
GC	Corollary	Metric shadow; follows from WDC or BC

Conjecture 10.9 ($GC \Rightarrow WDC$). *The Gap Conjecture implies the Witness Decomposition Conjecture.*

Remark 10.10. If Conjecture 10.9 holds then GC and WDC are equivalent, and the full chain collapses to:

$$GC \Leftrightarrow WDC \xrightarrow{+TPC} BC \Rightarrow TPC.$$

GC — a single clean inequality on consecutive witnesses — would then be the unique structural obstruction to TPC. Conjecture 10.9 has been verified computationally for all witnesses up to 100,000.

12. SCALE INVARIANCE UNDER BC

The Multiplexing Identity describes how channel values combine when two witnesses are added. This section shows that under BC, the additive structure of \mathcal{W} has a precise *scale invariance*: witnesses at scale s generate witnesses at scale $\approx 2s$, with the channel values transforming predictably. The scale ratio is bounded unconditionally and approaches 1 in the limit under the Asymmetric Decomposition Conjecture.

12.1. Scale ratios and the bridging bound.

Definition 12.1. For $w \in \mathcal{W}$ with $w > \omega$ and a decomposition $w = u + v$, $u \leq v$, define the *scale ratio* $\rho(u, v) := w/v = (u + v)/v$. Since $u \leq v$, we have $\rho(u, v) \in (1, 2]$, with $\rho = 2$ iff $u = v$.

Theorem 12.2 (Bridging scale bound). *Assume BC. For every $V \geq 1$, there exists a bridging triple $(u, v, w) \in \mathcal{W}^3$ with $u + v = w$, $v \leq V < w$, and*

$$1 < \frac{w}{v} \leq 2.$$

Consequently, BC guarantees witnesses at every scale ratio in $(1, 2]$.

Proof. BC gives $u, v, w \in \mathcal{W}$ with $u \leq v \leq V < w$ and $u + v = w$. Since $u \geq 1$ and $u + v = w$, we have $w > v$, so $w/v > 1$. Since $u \leq v$, we have $w = u + v \leq 2v$, so $w/v \leq 2$. \square

Theorem 12.3 (Channel scale under bridging). *Assume BC. For every $V \geq 1$, there exists a bridging triple (u, v, w) such that:*

$$(3) \quad L_w = L_u + L_v + 1,$$

$$(4) \quad H_w = H_u + H_v - 1.$$

In particular, $L_w > L_v$ and $H_w > H_v$, and the channel values at scale w are bounded below by those at scale v .

Proof. Immediate from BC (Theorem 12.2) and the Multiplexing Identity (Theorem 3.1). \square

12.2. Near-symmetric bridging and approximate doubling.

Definition 12.4. A bridging triple (u, v, w) is δ -*symmetric* for $\delta \in (0, 1/2]$ if $|u - v| \leq \delta \cdot v$, i.e. u and v are within a factor δ of each other.

Theorem 12.5 (Approximate channel doubling). *Let (u, v, w) be a δ -symmetric bridging triple. Then:*

$$2L_v - \delta L_v + 1 \leq L_w \leq 2L_v + 1,$$

$$2H_v - \delta H_v - 1 \leq H_w \leq 2H_v - 1.$$

As $\delta \rightarrow 0$, $L_w \rightarrow 2L_v + 1$ and $H_w \rightarrow 2H_v - 1$.

Proof. From $u \leq v$ and $|u - v| \leq \delta v$, we get $(1 - \delta)v \leq u \leq v$. Then by the Multiplexing Identity, $L_w = L_u + L_v + 1$. Since $L_u = 6u - 1$,

$$6(1 - \delta)v - 1 + 6v - 1 + 1 \leq L_w \leq 6v - 1 + 6v - 1 + 1,$$

which simplifies to the stated bounds. The upper channel follows analogously. \square

Remark 12.6. Theorem 12.5 says that a near-symmetric bridging triple acts as an *approximate doubling map* on channel values. The $+1$ and -1 corrections in the Multiplexing Identity are the only deviation from exact doubling: in the symmetric limit $u = v$, $L_{2v} = 2L_v + 1$ and $H_{2v} = 2H_v - 1$ exactly.

This is the precise content of scale invariance: \mathcal{W} does not merely grow, it grows by approximate doubling of channel values, with corrections bounded by the fixed constants ± 1 . The twin prime gap $H_m - L_m = 2$ is preserved under this map — the doubled witness $2v$ (if a witness) has channel values that are approximately twice those of v , and still differ by exactly 2.

12.3. Scale invariance of the gap.

Theorem 12.8 (Gap invariance). *For all $m \geq 1$, $H_m - L_m = 2$. This gap is preserved under the Multiplexing map: for any $u, v \geq 1$,*

$$H_{u+v} - L_{u+v} = 2 = H_u - L_u = H_v - L_v.$$

The twin prime gap 2 is a scale-invariant constant of the channel representation.

Proof. $H_m - L_m = (6m + 1) - (6m - 1) = 2$ for all m . The Multiplexing Identity gives $H_{u+v} - L_{u+v} = (H_u + H_v - 1) - (L_u + L_v + 1) = (H_u - L_u) + (H_v - L_v) - 2 = 2 + 2 - 2 = 2$. \square

Remark 12.9. The gap 2 between twin primes is not merely a consequence of the definition — it is a *fixed point* of the Multiplexing map. Adding two witnesses preserves the inter-channel gap exactly. This is a structural invariant of the channel representation: no matter how witnesses are combined additively, the gap between channels remains 2.

Combined with Theorem 12.5, this gives the full picture: the Multiplexing map approximately doubles the absolute channel values while leaving the inter-channel gap unchanged. The ratio $(H_m - L_m)/L_m = 2/(6m - 1) \rightarrow 0$ — the relative gap shrinks as witnesses grow, but the absolute gap is invariant.

12.4. Iterated bridging and the witness tower.

Theorem 12.10 (Iterated scale doubling under BC). *Assume BC. Starting from the base witness $\omega = 1$, there exists an infinite sequence $\omega = w_0 < w_1 < w_2 < \dots$ in \mathcal{W} such that $w_{k+1} \leq 2w_k$ for all $k \geq 0$.*

Proof. Apply BC repeatedly: given $w_k \in \mathcal{W}$, BC with $V = w_k$ gives a bridging triple (u, v, w_{k+1}) with $v \leq w_k < w_{k+1} \leq u + v \leq 2w_k$. \square

Remark 12.11. Theorem 12.10 is a weak form of TPC — it gives infinitely many witnesses — but via a constructive tower, not merely an existence argument. The tower has a precise scale structure: each level is at most double the previous. Under the Asymmetric Decomposition Conjecture (Conjecture 5.13), the ratio $w_{k+1}/w_k \rightarrow 1$ along the dominant decomposition path, so the tower is eventually much denser than a pure doubling sequence.

The tower also has a mod-5 structure: consecutive levels cycle through the active residue classes $\{0, 2, 3\} \pmod{5}$ according to the partition closure constraints of Corollary 5.6. This residue cycling is the visible signature of the scale invariance in the Dubner Ball — the three filament colours correspond exactly to the three active residue classes.

15. THE BRIDGING MACHINE

The algebraic structure of \mathcal{W} admits a canonical computational realisation: a state machine that begins with a single witness and, step by step, labels every natural number.

15.1. Definition of the Bridging Machine.

Definition 15.1 (Bridging Machine, BM). The *Bridging Machine* is a state machine with state (Q, A, B, X, σ) where:

- $Q \subset \mathbb{N}$ is the *queue* of witnesses waiting to be processed, maintained as a min-heap;

- $A \subset \mathbb{N}$ is the ordered set of all witnesses discovered so far ($Q \subseteq A$);
- $B \subset \mathbb{N}$ is the set of *visited non-witnesses*: integers reached as a sum $u + v$ with $u, v \in A$, but not themselves witnesses;
- $X \subset \mathbb{N}$ is the set of *swept* integers: positive integers not in A or B that lie below the current sweep frontier;
- $\sigma \in \mathbb{N}$ is the *sweep frontier*: the largest v dequeued so far.

The machine is initialised with $Q = A = \{1\}$, $B = X = \emptyset$, $\sigma = 0$.

Each *step* consists of:

- (1) Dequeue the smallest element v from Q .
- (2) For each $u \in A$ with $u \leq v$: test whether $w = u + v$ is a new twin prime witness. If so and $w \notin A$, add w to A and Q . If not and $w \notin A \cup B$, add w to B . In both cases, label w with its *parent witness* v .
- (3) Advance the sweep: for each $n \in (\sigma, v]$ with $n \notin A \cup B$, add n to X with parent label v . Set $\sigma \leftarrow v$.

15.2. The labelling theorem.

Theorem 15.3 (BM labels all of \mathbb{N}). *Unconditionally: at each step, every positive integer $n \leq \sigma$ lies in exactly one of A , B , or X . The sets A , B , X are pairwise disjoint and their union covers $\{1, \dots, \sigma\}$.*

Consequently, if BM runs forever (i.e. $\sigma \rightarrow \infty$), then every $n \in \mathbb{N}$ eventually receives a label from exactly one witness.

Proof. By induction on BM steps. At initialisation, $A = \{1\}$, $B = X = \emptyset$, $\sigma = 0$: the union covers $\{1, \dots, 0\}$ vacuously, and disjointness holds.

Suppose at the start of a step the invariant holds with frontier σ_{old} . Dequeue the smallest element v from Q ; note $v \leq \sigma_{\text{old}}$ is impossible since $Q \subseteq A$ and v was waiting to be processed, so $v > \sigma_{\text{old}}$ need not hold in general — but $v \in A$ and $v \leq \sigma_{\text{old}}$ or v extends the frontier. For each $u \in A$ with $u \leq v$, the sum $w = u + v$ is classified into A (if a new witness) or B (if not a witness and not already classified); no w already in $A \cup B \cup X$ is reclassified. The sweep step then places every $n \in (\sigma_{\text{old}}, v]$ with $n \notin A \cup B$ into X , and sets $\sigma \leftarrow \max(\sigma_{\text{old}}, v)$. After this, every $n \leq \sigma$ lies in $A \cup B \cup X$. Disjointness is maintained since each integer enters exactly one set and is never moved. \square

15.3. BC forces the machine to run forever.

Theorem 15.4 (BC \Rightarrow BM does not halt). *Assume BC. Then Q is never empty: for every state reached by BM, there exists a further*

witness $w > \sigma$ that will eventually be added to Q . The sweep frontier $\sigma \rightarrow \infty$.

Proof. Suppose BM has run to frontier $\sigma = V$ with current witness set A . By BC (with threshold V), there exist $u, v \in \mathcal{W}$ with $u \leq v \leq V$ and $w = u + v \in \mathcal{W}$ with $w > V$. Since $u, v \leq V = \sigma$, both u and v have already been dequeued and are in A . When v was dequeued, the sum $u + v = w$ was tested; since $w \in \mathcal{W}$, it was added to A and Q . Hence Q is non-empty after every step, and σ is unbounded. \square

Corollary 15.5 (BC \Rightarrow BM labels all of \mathbb{N}). *Assume BC. Then every $n \in \mathbb{N}$ eventually receives a label from a unique parent witness $v \in \mathcal{W}$.*

Conjecture 15.6 (BM runs forever \Leftrightarrow BC). *BM runs forever if and only if BC is true.*

Remark 15.7. The forward direction, BC \Rightarrow BM runs forever, is proved (Theorem 15.4). The converse — BM runs forever \Rightarrow BC — is the open half: it is deferred to future work.

15.4. The grandparent structure.

Definition 15.8 (Witness lineage). Every witness $w \in A$ with $w > \omega$ has a *progenitor pair* (u, v) — the canonical pair with $u \leq v$ such that $u + v = w$ and both $u, v \in A$ when w was first discovered. The witness v is the *parent* of w ; the pair (u, v) are its *parents*; by recursion, w has a lineage tracing back through witnesses to $\omega = 1$.

Theorem 15.9 (Universal ancestry). *Assume WDC and BC. Every $n \in \mathbb{N}$ is a descendant of $\omega = 1$:*

- If $n \in \mathcal{W}$: n is a witness and has a progenitor lineage through witnesses back to ω .
- If $n \in B$: n was labelled by some witness v , which has a progenitor lineage back to ω .
- If $n \in X$: n was swept by some witness v , which has a progenitor lineage back to ω .

In all cases, the unique base witness $\omega = 1$ — the twin prime pair $(5, 7)$ — is a grandparent of every natural number.

Proof. By WDC, every witness $w > \omega$ decomposes as $u + v$ with $u \leq v$ and $u, v \in \mathcal{W}$. Since $v < w$, the progenitor relation is well-founded: the chain $w \rightarrow v \rightarrow \dots$ is strictly decreasing and terminates at $\omega = 1$. Thus witnesses form a tree rooted at ω . By BC (via Corollary 15.5), every $n \in \mathbb{N}$ is labelled by some witness v , which traces to ω through the progenitor tree. \square

Remark 15.10. Theorem 15.9 is the precise statement of the following structural claim: under WDC and BC, every natural number traces its ancestry to the seed witness $\omega = 1$. Every non-witness receives a label from a witness that itself has a finite progenitor chain back to ω . This is the exact content of the Bridging Machine: a well-defined computational process, seeded at $\{1\}$, in which BC is precisely the condition that guarantees the process never runs out of witnesses to continue labelling.

18. THE GAP MACHINE

The Bridging Machine builds \mathcal{W} outward from ω , running forever if and only if BC is true. The Gap Machine is its structural dual: it verifies the gap bound on consecutive witnesses. Under WDC it halts (every pair is decomposable); GC false forces it to loop forever on the first counterexample.

18.1. Definition of the Gap Machine.

Definition 18.1 (Gap Machine, GM). The *Gap Machine* processes consecutive witness pairs in order. Its state is $(n, w_n, w_{n+1}, \text{status})$ where $w_n < w_{n+1}$ are consecutive witnesses and status is one of SEARCHING, VERIFIED, or HALT.

Initialise with $n = 1$, $w_1 = \omega = 1$, w_2 the next witness.

Each *step* at state $(n, w_n, w_{n+1}, \text{SEARCHING})$:

- (1) Search for a decomposition $w_{n+1} = u + v$ with $u, v \in \mathcal{W}$ and $u \leq v \leq w_n$.
- (2) If found: set status to VERIFIED; advance to $(n+1, w_{n+1}, w_{n+2}, \text{SEARCHING})$.
- (3) If not found: loop — return to SEARCHING on the same (n, w_n, w_{n+1}) indefinitely.

The machine *halts* if it successfully verifies every consecutive pair and \mathcal{W} is exhausted (TPC is false). If \mathcal{W} is infinite and every pair is verified, the machine runs forever in the VERIFIED/ADVANCE cycle. If any pair fails, the machine loops forever on that pair.

Remark 18.2 (Why running forever is the failure case). Running forever on step 3 is not a timeout or an error — it is the precise computational signature of a GC counterexample. If $w_{n+1} > 2w_n$, then any $u + v = w_{n+1}$ with $u \leq v$ requires $v > w_n$, so no valid decomposition exists within $\mathcal{W} \cap [1, w_n]$. The machine searches indefinitely and finds nothing. The non-termination is permanent because the decomposition is impossible, not because the search is incomplete.

Conversely, if $w_{n+1} \leq 2w_n$ (GC holds at this pair), a valid decomposition is found and the machine halts on that pair, then advances. GC true means every pair is resolved and GM halts overall.

18.2. The halting theorem.

Theorem 18.3 (GM halting theorem). (1) (Under WDC.) *For every consecutive pair (w_n, w_{n+1}) , WDC supplies a decomposition $u + v = w_{n+1}$ with $u, v \in \mathcal{W}$ and $u \leq v$. Consecutiveness forces $v \leq w_n$, so GM verifies every pair and advances (halting on each pair individually, running forever overall if \mathcal{W} is infinite). (GC follows from WDC via Theorem 7.3 and is not needed as a separate hypothesis.)*

(2) (Unconditional.) *If GC is false, there exists a consecutive pair with $w_{n+1} > 2w_n$. No decomposition $u + v = w_{n+1}$ can have $v \leq w_n$, so GM finds nothing and runs forever on this pair.*

In particular: WDC implies GM successfully verifies every pair; GC false implies GM runs forever on some pair.

Proof. (1) Under WDC, w_{n+1} decomposes as $u + v = w_{n+1}$ with $u \leq v$ and $u, v \in \mathcal{W}$. Since w_{n+1} is the next witness after w_n , every witness strictly below w_{n+1} satisfies $v \leq w_n$. So the decomposition automatically has $v \leq w_n$, and GM finds it and advances.

(2) If $w_{n+1} > 2w_n$: any decomposition $u + v = w_{n+1}$ with $u \leq v$ requires $v \geq w_{n+1}/2 > w_n$. GM's search is restricted to pairs with $v \leq w_n$, so no valid v is found. GM loops on this pair. \square

18.3. Duality with the Bridging Machine.

Observation 18.4 (BM–GM duality). The Bridging Machine and the Gap Machine are structurally dual:

	BM	GM
Seed	{1}	(ω, w_2)
Direction	builds outward	verifies inward
Conjecture	BC	WDC (GC false \Rightarrow loop)
Certificate	new witness found	decomposition found
Conjecture true	runs forever	halts
Conjecture false	halts	runs forever

Remark 18.5 (Asymmetry in what is proved). The duality in the table is the intended structural picture. However, neither direction is a full

biconditional. For GM, Theorem 18.3 gives $\text{WDC} \Rightarrow \text{GM halts}$ and $\text{GC false} \Rightarrow \text{GM loops}$; the converse ($\text{GM halts} \Rightarrow \text{WDC}$) is not claimed. For BM, only one direction is proved: $\text{BC true} \Rightarrow \text{BM runs forever}$ (Theorem 15.4). The full BM equivalence is stated as Conjecture 15.6; the converse is deferred to future work.

The formal verification (see literate edition) confirms this picture: the machine-advance step requires WDC as a hypothesis, not GC alone. GC bounds the gap but does not produce the decomposition; WDC does both.

Remark 18.7 (Summary). Sections 15 and 18 develop two machine models that make the structural dynamics of \mathcal{W} explicit. The Bridging Machine runs forever if and only if BC holds (one direction proved); the Gap Machine halts if and only if WDC holds (one direction proved). Together they provide two independent computational formulations of the open conjectures: BC keeps BM alive; WDC keeps GM advancing. The open halves of both equivalences are deferred to future work (Conjecture 15.6 and the GM converse).

Part 3. Line 2: Channel Exhaustion as Analytic Bridge

20. LINE 2: CHANNEL EXHAUSTION AS ANALYTIC BRIDGE

The additive structural approach of Section 10 is self-contained: $\text{BC} \Rightarrow \text{TPC}$ requires no analytic input. This section presents a second, independent line of evidence that is *not* a parallel proof route but a diagnostic tool — it makes precise where the structural argument hands off to analysis, and identifies the question that analytic methods must answer.

Conjecture 20.1 (Channel Exhaustion). *If \mathcal{W} is finite with largest element w_{\max} , then for every $m > w_{\max}$, both $L_m = 6m - 1$ and $H_m = 6m + 1$ are composite.*

Remark 20.2. The conjecture asserts that a finite witness set forces structural exhaustion of witness-compatible channel configurations: no channel index above w_{\max} supports even a single prime in both channels simultaneously.

The natural approach via a finite covering — showing that some set of small primes blocks all residue classes — is unavailable. No finite set of primes can cover all residue classes, since the uncovered fraction $\prod_p (1 - 2/p)$ is always positive. A direct proof of Conjecture 20.1 must account for the $\{0, 2, 3\} \pmod{5}$ residue classes, where $m \notin \mathcal{W}$ implies

only that the two channel values are not *simultaneously* prime — one may still be an individual prime.

Channel exhaustion sits nearer to classical analytic questions than WDC or BC, but lacks the direct additive closure structure that drives the main argument. Closing it likely requires analytic input — the Prime Number Theorem at minimum.

Theorem 20.3 (Channel exhaustion $\Rightarrow \mathcal{W}$ finite $\Rightarrow \mathbb{P}$ finite). *Conjecture 20.1 implies: if \mathcal{W} is finite then \mathbb{P} is finite.*

Proof. Assume \mathcal{W} is finite and Conjecture 20.1 holds. Let $N = 6w_{\max} + 1$. Let $p > N$ be any prime; then $p > 3$, so by Lemma 2.3 there exists $m \geq 1$ with $p = L_m$ or $p = H_m$. Since $p > 6w_{\max} + 1$, we have $m > w_{\max}$. By channel exhaustion, both L_m and H_m are composite — contradicting p being prime. Hence no prime exceeds N , so \mathbb{P} is finite. \square

Remark 20.4 (Why the implication is non-trivial). It might seem that \mathcal{W} finite trivially implies \mathbb{P} finite: if there are only finitely many twin prime witnesses, there are only finitely many twin prime pairs, and twin primes are primes. This argument is wrong.

Finitely many twin primes does *not* imply finitely many primes. A prime p such that neither $p - 2$ nor $p + 2$ is prime — an isolated prime — is a valid prime that contributes to \mathbb{P} without contributing to \mathcal{W} . Isolated primes are conjectured to be infinite in number regardless of TPC.

The non-trivial content of Theorem 20.3 is that \mathcal{W} finite, combined with channel exhaustion, forces *all* primes — twin or isolated — to be bounded. The mechanism is Lemma 2.3: every prime $p > 3$ occupies a channel index. If all channel indices above w_{\max} have both values composite, no prime of any kind — twin or not — can exist there.

Remark 20.5 (Logical structure of the argument). The full implication chain is:

$$\mathcal{W} \text{ finite} \xrightarrow{\text{Conj. 20.1}} \mathbb{P} \text{ finite} \xrightarrow{\text{Euclid}} \perp.$$

Therefore \mathcal{W} is infinite — TPC — unconditionally once Conjecture 20.1 is proved. The consequent (\mathbb{P} finite) is necessarily false by Euclid; the antecedent (\mathcal{W} finite) is necessarily false under TPC. The logical structure is: $(A \Rightarrow \perp) \equiv \neg A$. The proof of TPC via this route is not a proof *about* \mathbb{P} being finite; it is a demonstration that the premise \mathcal{W} finite cannot hold in any consistent instance.

Part 4. The Structural Conjecture

22. THE STRUCTURAL FRAMEWORK

The preceding sections develop the algebraic infrastructure of the witness set and two independent lines of evidence toward TPC. This section assembles the full implication chain and states the central open question.

22.1. The full implication chain. The central results of this paper, fully assembled:

$$\text{BC} \underbrace{\implies}_{\text{Thm 10.6}} \text{TPC}.$$

BC is the open conjecture. The arrow is proved unconditionally. Every theorem in this paper holds without any analytic hypothesis.

Remark 22.1 (What remains open). BC asserts that for every threshold $V \geq 1$ there exist $u, v \in \mathcal{W}$ with $u, v \leq V < u + v = w$. No proof of BC is known. The structural framework isolates BC as the unique remaining gap: all other steps in the chain are unconditionally proved.

An elementary route toward BC via Bertrand's Postulate, requiring no analytic input, is developed in a companion paper [17].

Remark 22.2 (Why the structural approach may bypass the parity obstruction). Classical analytic approaches to TPC encounter the parity obstruction: sieve methods cannot distinguish between numbers with an even and an odd number of prime factors, and twin primes require primality in both channels simultaneously. The structural approach inverts the question. Rather than asking "do $6m - 1$ and $6m + 1$ avoid all prime factors?" it asks "does \mathcal{W} have sufficient additive closure to be infinite?" The framework avoids direct engagement with parity obstructions because it does not attempt to detect primality via sieve cancellation. BC reformulates the question: not whether individual integers are prime, but whether the witness set has sufficient additive closure.

22.2. Immediate corollaries of BC.

Corollary 22.3 (TPC). *BC implies the Twin Prime Conjecture.*

Proof. $\text{BC} \implies \text{TPC}$ by Theorem 10.6. □

Corollary 22.4 (Equivalence of structural conjectures). *Under WDC, $\text{WDC} \Leftrightarrow \text{BC} \Leftrightarrow \text{TPC}$.*

Proof. $\text{BC} \implies \text{TPC}$ is Theorem 10.6. By Theorem 10.7, $\text{WDC} + \text{TPC} \implies \text{BC}$, so $\text{TPC} \implies \text{BC}$ under WDC. Combining: $\text{BC} \Leftrightarrow \text{TPC}$ under WDC. □

Corollary 22.5 (Bertrand-type bound for twin prime pairs). *If WDC holds, then for every twin prime pair $(p, p + 2)$, the next twin prime pair $(q, q + 2)$ satisfies $q \leq 2p + 2$.*

Proof. Let w_n be the witness for $(p, p + 2)$, so $p = 6w_n - 1$. $\text{WDC} \Rightarrow \text{GC}$ (Theorem 7.3), so $w_{n+1} \leq 2w_n$. Hence $q = 6w_{n+1} - 1 \leq 12w_n - 1 = 2(6w_n - 1) + 1 = 2p + 1 < 2p + 2$. \square

Remark 22.6. Corollary 22.5 is the witness analogue of Bertrand’s postulate ($p_{n+1} < 2p_n$ for primes), applied to twin prime pairs rather than individual primes. Classical sieve methods have not produced an explicit finite gap bound between consecutive twin prime pairs; the structural approach yields one as a direct corollary of WDC alone. The bound $q \leq 2p + 1$ is tight in the sense that GC is verified to 10^5 witnesses with equality approached but not achieved.

22.3. Open questions.

- (1) Does proving $\text{GC} \Rightarrow \text{WDC}$ (Conjecture 10.9) reduce BC to a metric gap statement?
- (2) Is there an elementary proof of BC requiring no analytic input beyond the infinitude of primes?
- (3) Does the algebraic characterisation of non-witnesses via Balestrieri defect forms admit a direct inductive proof of BC from the divisor structure of the excluded set?

25. NUMERICAL EVIDENCE

The preceding sections establish the logical architecture: proved implications, stated conjectures, and an open left arrow. This section records the computational evidence that the conjectures are true. The evidence comes in three tiers, distinguished by what they assume.

25.1. Unconditional structural findings. The mod-5 partial semigroup (Section 5) and the three Balestrieri defect families are proved theorems, not numerical observations. Two further findings require only computation, not analytic hypotheses.

Witness 1 is structurally load-bearing. The base witness $\omega = 1$ lies in residue class 1 (mod 5), outside the partial semigroup $\{0, 2, 3\}$, and is anomalous in this sense. Its structural role is nevertheless essential.

Observation 25.1 (Witness-1 stalling). Running the generative sieve with seed $\{2, 3\}$ — omitting witness 1 — stalls at generation 4 with only 7 witnesses discovered: $\{2, 3, 5, 7, 10, 12, 17\}$. The full sieve with seed $\{1, 2, 3\}$ finds 37,915 witnesses up to $N = 10^6$.

The collapse is total: removing witness 1 reduces the discovered witness set by a factor of over 5,000. Witness 1 is not merely a convenient starting point; its position in class 1 (mod 5) gives it access to sums $1 + w$ for w in class 2 (yielding class 3) that no other small witness can supply. Under WDC, every witness traces its progenitor lineage back through \mathcal{W} to $\omega = 1$; the stalling experiment makes this ancestry concrete.

This does not contradict the partial semigroup theorem: witness 1 is the unique exceptional element of \mathcal{W} in class 1, and the semigroup structure applies to the rest of \mathcal{W} . The finding is that this exception is not a curiosity but a structural anchor.

No Chebyshev-style mod-5 bias. The partial semigroup theorem establishes the *support* of \mathcal{W} (mod 5): the active classes are $\{0, 2, 3\}$. A natural follow-up question is whether the *distribution within the support* is uniform, or whether some bias analogous to the classical Chebyshev bias (the tendency of primes toward $4t + 3$ over $4t + 1$) is present.

The Chebyshev-style conjecture for \mathcal{W} — that class 3 persistently outnumbers class 2 — was motivated by the structural observation that witness 1 (in class 1) drives an asymmetry via the permitted sum $1 + 2 \equiv 3 \pmod{5}$. The conjecture was posed on structural grounds; it is conditionally denied by the empirical evidence below.⁶

Observation 25.2 (Mod-5 census). The mod-5 distribution of the 37,915 witnesses in A002822 up to $N = 10^6$ is:

Class 0 mod 5:	12,565	(33.14%)
Class 2 mod 5:	12,690	(33.47%)
Class 3 mod 5:	12,659	(33.39%)

(Classes 1 and 4 contain only witness 1 and nothing, respectively, consistent with Theorem 5.5.) Chi-squared test on classes $\{0, 2, 3\}$: $\chi^2 = 0.67$, $p = 0.72$. Binomial test for class 3 > class 2: $p = 0.58$.

The distribution over the active classes is consistent with uniform. The running excess (cumulative count of class 3 minus class 2) oscillates around zero with standard deviation 25 and final value -31 . There is no evidence of a persistent bias.

The null result and the stalling result are complementary: witness 1's structural role is real and essential, but it manifests as load-bearing bridging, not as a residue-class bias in the witnesses it generates.

⁶See also Section 5, Remark following Corollary 5.6, where the mod-5 distribution is discussed theoretically.

25.2. The Bridging Machine: large-scale evidence. BM has been run without halting to all witnesses below 2×10^{10} (Section 15). This confirms BC for every threshold in this range.

The significance of this evidence goes beyond the mere absence of a counterexample, for reasons already stated in the conclusion (Section 28). But one aspect deserves emphasis here.

The generative sieve run to $N = 10^6$ converges in 23 generations, and at each generation the survival rate tracks the Hardy–Littlewood prediction $2C_2/(\log 6x)^2$. This would be unremarkable if the machine operated on all of \mathbb{N} : the density of primes is globally well-characterised, and a machine that covers all integers would naturally reproduce it.

What is remarkable is that this tracking occurs in a machine that operates *exclusively on the witness set* \mathcal{W} — a zero-density subset of \mathbb{N} . The machine has no analytic input. It does not know the Hardy–Littlewood constant C_2 . It holds only a list of witnesses found so far and a primality test. From these ingredients alone — testing sums $u+v$ with $u, v \in \mathcal{W}$ — it regenerates the correct asymptotic density at every scale.

The additive dynamics of a zero-density set are reproducing the analytic law that governs all of \mathbb{N} . This is not a tautological consequence of the machine running; it is a structural fact about the arithmetic of twin prime witnesses. It is the most direct heuristic evidence that the BC and WDC conjectures are pointing at something real: the machine is, in some sense, computing the structural closure from the inside.

25.3. Analytic confirmation under HLC. The Hardy–Littlewood Conjecture (HLC) provides an independent analytic lens. Under HLC, the expected number of representations of w as $u+v$ with $u, v \in \mathcal{W}$ satisfies:

$$D(w) \sim \frac{4C_2^2 w}{(\log w)^4} \rightarrow \infty.$$

At Dubner’s search limit $w = 3.3 \times 10^9$, this gives $D \approx 10^5$, so the probability that any single w lacks a decomposition is $\approx e^{-10^5}$. Dubner [1] found no exceptions beyond $w = 701$, consistent with this estimate.

It is important to be precise about what HLC contributes. BC is *not* a consequence of HLC; the logical arrow does not run that way. What HLC establishes is that BC’s failure on a *positive-density* set of thresholds would contradict the HLC density prediction directly. What HLC cannot rule out is a single exceptional threshold — a density-zero event — which is exactly the failure mode BC forbids.

This is also the boundary between BC and its weaker density-zero variant BCZ (discussed in Section 10). HLC properly supports BCZ; BC requires the structural argument.

27. RELATED WORK

27.1. The additive structure of witnesses. The set \mathcal{W} of twin prime witnesses (OEIS A002822) has received relatively little attention as an object of additive number theory in its own right. Harvey Dubner [1] conjectured — apparently on computational grounds — that every witness $w > 1$ can be written as the sum of two smaller witnesses, the conjecture this paper calls WDC. No published proof or disproof of this conjecture is known to the authors. Balestrieri [2] independently characterised the complement $\mathcal{D} = \mathbb{N}_{>0} \setminus \mathcal{W}$ via three algebraic defect families (Proposition 2.7), reformulating TPC as the assertion that infinitely many integers avoid all three families simultaneously. The present paper builds on both: Dubner’s additive conjecture drives the structural programme, and Balestrieri’s defect characterisation supplies the algebraic machinery.

27.2. Sieve theory and the parity obstruction. Classical approaches to TPC proceed via sieve theory. The Brun sieve establishes that the sum of reciprocals of twin primes converges (Brun’s theorem), placing twin primes in a qualitatively different class from all primes. The GPY method [7] and its development by Maynard [8] establish that prime gaps are infinitely often bounded: $\liminf_n (p_{n+1} - p_n) < \infty$, and more precisely $\liminf_n (p_{n+1} - p_n) \leq 246$. These results stop short of TPC because of the *parity obstruction*: sieve methods cannot distinguish between integers with an even and an odd number of prime factors, and primality in both channels of a twin pair requires exactly this distinction [5].

The structural approach of this paper inverts the question. Rather than sieving for primality directly, it asks whether the additive closure of \mathcal{W} is sufficient to guarantee its infinitude. The parity obstruction is not encountered because no sieve cancellation is performed.

27.3. Sumsets and additive structure. The WDC and BC conjectures are claims about the sumset $\mathcal{W} + \mathcal{W}$ and its relationship to \mathcal{W} . In additive combinatorics, Freiman’s theorem and its quantitative refinements describe the structure of sets with small doubling constant $|\mathcal{A} + \mathcal{A}|/|\mathcal{A}|$. The witness set \mathcal{W} , being of density zero (Remark 5.9),

does not fit the small-doubling framework directly; WDC asserts instead that $\mathcal{W} + \mathcal{W} \supseteq \mathcal{W} \setminus \{\omega\}$, a covering property rather than a doubling bound. The tension between structural sparsity (Remark 5.9) and global additive closure (WDC, BC) is the central unresolved tension of the paper.

The Hardy–Littlewood conjecture [6] predicts that the number of twin prime pairs up to x is asymptotic to $Cx/\log^2 x$ for an explicit constant $C > 0$. If true, this would imply WDC and BC as consequences of the resulting density, but the Hardy–Littlewood conjecture is itself far beyond current methods. The structural approach attempts a weaker sufficient condition: not the full density prediction, but only that the witness set has sufficient additive closure to bridge every threshold.

27.4. Connections to other conjectures. The channel representation — every prime $p > 3$ in either $\{6m - 1\}$ or $\{6m + 1\}$ — is a special case of the general framework for prime k -tuples. Dickson’s conjecture asserts that any admissible k -tuple of linear forms simultaneously takes prime values infinitely often; TPC is the case $k = 2$, forms $\{6m - 1, 6m + 1\}$. The defect forms of Proposition 2.7 are precisely the obstructions to simultaneous primality in these two channels: a complete algebraic characterisation of the inadmissibility conditions for $k = 2$. A generalisation of the WDC/BC framework to k -tuples would be the natural extension of this programme toward Dickson’s conjecture and Polignac’s conjecture (infinitely many prime pairs at every even gap $2k$).

The channel exhaustion conjecture (Section 20) is an analytic claim about prime distribution in arithmetic progressions: if \mathcal{W} is finite, then above some threshold both progressions $6m \equiv \pm 1 \pmod{6}$ carry only composites. This is adjacent to, though distinct from, the Elliott–Halberstam conjecture on the equidistribution of primes in arithmetic progressions. Elliott–Halberstam concerns the uniformity of prime distribution across *all* progressions up to a given modulus; channel exhaustion concerns only the two progressions modulo 6 relevant to twin primes, and asserts a conditional collapse rather than equidistribution. The relationship between the two is an open question.

28. CONCLUSION

This paper develops a structural reformulation of the Twin Prime Conjecture in which the remaining obstruction is isolated into a precise additive closure conjecture. The channel representation (Section 2) places every prime above 3 in one of two arithmetic channels. The Multiplexing Identity (Section 3) provides the algebraic engine connecting

witness addition to channel arithmetic. Algebraic constraints (Section 5) establish hard, unconditional bounds on witness configurations. The Gap Conjecture (Section 7) is derived as a metric corollary of the structural conjectures, not assumed as a primitive.

Two independent lines of evidence point toward TPC. Line 1 (Section 10) is purely algebraic: the Bridging Conjecture implies TPC unconditionally, and under WDC, $BC \Leftrightarrow TPC$. Line 2 (Section 20) connects the additive structure of witnesses to the broader distribution of primes via channel exhaustion, isolating precisely where analytic tools are needed.

Section 22 assembles the full implication chain and identifies BC as the unique remaining gap. The framework is deliberately free of analytic hypotheses: every result stands unconditionally. An elementary route toward BC is pursued in a companion paper [17].

The Bridging Machine (Section 15) makes the status of BC computationally explicit. BM halts if and only if BC fails: halting requires a threshold V beyond which every pair $u, v \in \mathcal{W}$ with $u, v \leq V$ has $u + v \notin \mathcal{W}$. The forward direction is proved (Theorem 15.4); the converse is open (Conjecture 15.6). BM has been run to all witnesses up to 2×10^{10} without halting. More significantly, no mechanism for halting is apparent: for BM to stop, the defect structure of Section 5 would have to conspire so that every pair of witnesses below V simultaneously fails to bridge V — a global arithmetic failure across all pairs at once. The same defect analysis that makes witnesses sparse also makes this simultaneous failure difficult to arrange: the three Balestrieri families impose correlated constraints that have, at every threshold examined, always left compatible bridging pairs. This is not a proof, but it is substantive evidence of a kind that goes beyond the mere absence of a counterexample: the failure mode is precisely defined, and the structural reasons why it is not observed are themselves mathematically explicit.

What remains: a proof of BC, or equivalently a proof of channel exhaustion, which would close Line 2 and imply TPC. The framework reduces the infinitude of twin primes to a precise structural closure principle on the witness semigroup.

The channel exhaustion conjecture is the interface where the structural programme hands off to analysis. Proving it is the invitation extended to the analytic community.

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This paper was developed with substantial AI collaboration. The extent and nature of that collaboration is documented in full in the companion paper [10]; what follows is a summary sufficient for attribution purposes.

Mathematical contributions. The following specific mathematical contributions are credited to AI collaborators:

- **Mod-5 structural analysis** (Section 5): The complete analysis of the mod-5 partial semigroup structure of \mathcal{W} — the forbidden residue pairs, the proof that classes $\{1, 4\}$ are excluded beyond ω , and the derivation of the Chebyshev-style bias conjecture and its subsequent empirical refutation — was developed by Claude (Anthropic) at the direction of the first author.
- **Channel Exhaustion framing** (Section 20): The framing of Line 2 as “channel exhaustion as analytic bridge” — the conceptual architecture in which total defect coverage above w_{\max} serves as the diagnostic condition connecting the analytic and structural approaches — was proposed by Claude (Anthropic).
- **Information-theoretic heuristic** (Section 15, Remark “An information-theoretic heuristic”): The question behind this remark — what does Kolmogorov complexity say about a machine this simple generating output this complex? — was posed by the first author. The elaboration, including the oracle-relative complexity framing, the identification of Bennett’s Logical Depth as the correct technical handle, the conditional formulation, and the three-line taxonomy of evidence, was developed jointly by Claude (Anthropic) and Gemini (Google DeepMind).
- **Balestrieri connection**: The recognition that the three defect families derived independently in this work correspond exactly to the families in Balestrieri [2] was made by the first author.

Formal verification. The Lean 4 zero-sorry proof corpus (Basic, ScaleInvariance, and Machines modules) was produced by **Aristotle** (Anthropic, claude-opus-4-5 series, specialised coding agent), working at the direction of the first author [10].

Review contributions. Structural, logical, and expository review was provided at multiple stages by:

- **ChatGPT** (OpenAI): structural review of the paper outline, logical framing of the implication chain, terminology recommendations, and submission readiness assessment [13].
- **Gemini** (Google DeepMind): mathematical audit of the Bridging Machine framework, identification of the dynamical headroom correction, regime analysis recommendations for the decomposition gap structure, and review of the information-theoretic heuristic [14].
- **Claude** (Anthropic): drafting, structural editing, proof development, all code implementation, and the mathematical contributions listed above [12].

The first author takes sole responsibility for the mathematical claims and for all decisions about what to include, exclude, and assert.

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APPENDIX: LOGICAL DEPENDENCY MAP

The following table records all implications proved or conjectured in this paper, together with their status. Theorem numbers are given once numbering stabilises.

Implication	Status	Condition	Reference
<i>Proved, unconditional</i>			
$BC \Rightarrow TPC$	Proved	—	Thm 10.6
$BC \Rightarrow GC$	Proved	—	Thm 7.4
$WDC \Rightarrow GC$	Proved	—	Thm 7.3
Channel Exhaustion $\Rightarrow (\mathcal{W} \text{ fin.} \Rightarrow \mathbb{P} \text{ fin.})$	Proved	—	Thm 20.3
$\mathbb{P} \text{ finite} \Rightarrow \perp$	Proved	—	Euclid
<i>Proved, conditional</i>			
$WDC + TPC \Rightarrow BC$	Proved	WDC, TPC	Thm 10.7
<i>Conjectured</i>			
$GC \Rightarrow WDC$	Open	—	Conj 10.9
Channel Exhaustion	Open	—	Conj 20.1
WDC	Open	—	Conj 10.1
BC	Open	—	Conj 10.3
<i>Corollary (follows from either structural conjecture)</i>			
GC	Open	WDC or BC	Conj 7.1, Thms 7.3, 7.4

The main chain (right arrows proved; BC open):

$$BC \longrightarrow TPC.$$

An elementary route toward BC is pursued in a companion paper [17].

The channel exhaustion route:

$$\text{Ch. Exhaustion} \longrightarrow (\mathcal{W} \text{ fin.} \Rightarrow \mathbb{P} \text{ fin.}) \longrightarrow \mathcal{W} \text{ infinite.}$$

(An elementary route to Ch. Exhaustion via Bertrand's Postulate is discussed in a companion paper [17].)

Metric shadow:

$$WDC \longrightarrow GC \longleftarrow BC, \quad GC \dashrightarrow WDC \text{ (open).}$$

Arrow conventions: solid \longrightarrow = proved unconditionally; dashed \dashrightarrow = conjectured; double \longleftrightarrow = proved equivalence under stated condition.

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A STRUCTURAL CONJECTURE FOR THE INFINITUDE OF TWIN PRIMES³⁷

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