LOGICAL IMPLICATIONS OF A BRIDGING CONJECTURE FOR THE TWIN PRIME CONJECTURE

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Abstract. We introduce the Bridging Conjecture (BC) for the set $\mathcal{W}$ = A002822 of twin prime witnesses — positive integers w such that $6w-1$ and $6w+1$ are both prime — and study its logical relationship to the Twin Prime Conjecture (TPC) and to Dubner's Middle Number Conjecture (MNC) and its witness-set reformulation, the Witness Decomposition Conjecture (WDC). We prove two theorems. First, BC implies TPC unconditionally. Second, assuming WDC, BC and TPC are equivalent: WDC $\Rightarrow$ (BC $\Leftrightarrow$ TPC). Together these two theorems determine the logical landscape: of the eight combinations of truth values for WDC, BC, and TPC, exactly five are consistent with our results. We characterise each of the five possible worlds structurally. Supporting lemmas include a conditional gap bound $w_{n+1} - w_n \leq w_n$ and a lower bound $\pi_2(x) \gg \log x$, both following from BC alone. We also introduce the Middle Even Number Conjecture (MENC), which asserts that the exception set $\mathcal{X} = \mathbb{N}_{\geq 1} \setminus (\mathcal{W} + \mathcal{W})$ is finite; this is strictly weaker than Dubner's third conjecture (which pins $\mathcal{X}$ to a specific named set) but strictly stronger than WDC. We establish the implication chain Dubner 3 $\Rightarrow$ MENC $\Rightarrow$ WDC, and introduce density-zero variants BCZ, WDCZ, MENCZ with a parallel chain MENC $\Rightarrow$ MENCZ $\Rightarrow$ WDCZ and prove that WDCZ $\Rightarrow$ (BCZ $\Leftrightarrow$ TPC) (Theorem 12.4).

Part 1. Related Work

1. The Twin Prime Conjecture

A *twin prime pair* is a pair of primes $(p, p+2)$. The *Twin Prime Conjecture* asserts that there are infinitely many such pairs.

Conjecture 1.1 (Twin Prime Conjecture (TPC); folklore)**.** There are infinitely many primes p such that $p+2$ is also prime.

Date: May 2026.

The conjecture has been attributed to Euclid by tradition, though no explicit statement appears in his extant work; it is widely regarded as one of the oldest open problems in number theory.

Our terminology. The integer $p+1$ between a twin prime pair $(p, p+2)$ is divisible by 6; write $p+1 = 6w$, so w is a positive integer with $6w-1$ and $6w+1$ both prime. We call w a *twin prime witness* and write

$$\mathcal{W} = \{w \in \mathbb{N} : 6w-1 \text{ and } 6w+1 \text{ are both prime}\},$$

which is OEIS sequence A002822 [9]. In our terminology TPC is the assertion that $\mathcal{W}$ is infinite.

2. Oppermann's Conjecture

Conjecture 2.1 (Oppermann, 1882 [13])**.** For every integer $n > 1$, there is at least one prime in each of the intervals $(n(n-1),\, n^2)$ and $(n^2,\, n(n+1))$.

Oppermann's Conjecture implies that consecutive primes $p_k < p_{k+1}$ satisfy $p_{k+1} - p_k < \sqrt{p_k}$: the gap between consecutive primes is sub-square-root in scale.

Relevance to our work. In our terminology, Oppermann's Conjecture is a gap bound for the full prime sequence. The Bridging Conjecture (Section 6) is a structurally analogous gap bound for the twin prime witness sequence $\mathcal{W}$: it implies $w_{n+1} - w_n \le w_n$ (Lemma 6.8), placing BC in the same family as Oppermann's Conjecture but at a larger scale ($O(w_n)$ rather than $O(\sqrt{p_k})$) appropriate to the lower density of twin prime witnesses.

Oppermann's Conjecture remains open; the weaker Legendre variant (a prime between n^2 and $(n+1)^2$) has been verified computationally to $n \le 3.33 \times 10^{13}$ [14].

3. The Hardy–Littlewood Conjecture

Hardy and Littlewood [6] formulated a suite of quantitative conjectures on the distribution of prime constellations. The one most relevant here is their Conjecture B, which predicts the asymptotic density of twin prime pairs.

Conjecture 3.1 (Hardy–Littlewood Conjecture B [6])**.** The number of twin prime pairs $(p, p+2)$ with $p \le x$ is asymptotic to

$$\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2},$$

where $C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.6601\ldots$ is the twin prime constant.

The conjecture remains unproved. The best unconditional upper bound is Brun's theorem [5], which gives $\pi_2(x) = O(x(\log\log x)^2/(\log x)^2)$; this does not establish infinitude. Recent work of Zhang [7] and Maynard [8] establishes infinitely many prime pairs with bounded gaps, but does not yield an explicit lower bound on $\pi_2(x)$.

Our terminology. We write $|\mathcal{W} \cap [1, x]|$ for the witness counting function. Conjecture 3.1 implies $|\mathcal{W} \cap [1, x]| \sim C_2\, x/(\log x)^2$ (counting witnesses rather than pairs and adjusting constants). In Heuristic Evidence for the Bridging Conjecture [1] we use this asymptotic to estimate the expected number of additive decompositions of a witness and to argue that the sieve almost surely does not stall.

4. Dubner's Conjectures

In 1996, Stephen Wagler communicated a question to Paulo Ribenboim: what is the smallest middle number that is not the sum of two t-primes? Ribenboim asked Harvey Dubner to investigate [4, pp. 291–299]. Dubner's paper [3] formulated four conjectures, which we state here in his original terms and then restate in our terminology.

A *t-prime* is a prime belonging to a twin prime pair (OEIS A007534 [11]). A *middle number* is an integer of the form $p + 1$ where $(p, p + 2)$ is a twin prime pair; equivalently, $6w$ for some $w \in \mathcal{W}$.

Conjecture 4.1 (Dubner: Middle Number Conjecture (MNC) [3])**.** Every middle number greater than 6 is the sum of two middle numbers.

Since middle numbers are exactly the integers $6w$ for $w \in \mathcal{W}$, MNC is equivalent — dividing through by 6 — to the following reformulation in terms of witnesses.

Conjecture 4.2 (Witness Decomposition Conjecture (WDC))**.** Every witness $w > 1$ is the sum of two witnesses $u + v$ with $u, v \in \mathcal{W}$.

Conjecture 4.3 (Dubner: t-prime Sumset Conjecture [3])**.** Every middle number greater than 6 is the sum of two t-primes.

Conjecture 4.4 (Middle Even Number Conjecture (MENC))**.** The exception set $\mathcal{X} = \mathbb{N}_{\geq 1} \setminus (\mathcal{W} + \mathcal{W})$ is finite.

Conjecture 4.5 (Dubner: Exact Exception Set (Dubner 3) [3])**.** $\mathcal{X} =$ A243956; that is, $\mathcal{X}$ is exactly the twelve-element set $\{1, 16, 67, 86, 131, 151, 186, 191, 211, 226, 541$

Conjecture 4.6 (Dubner: t-prime Pair Sumset Conjecture [3])**.** Every even number greater than 4208 is the sum of two t-primes.

MENC is strictly stronger than WDC: requiring $\mathcal{X}$ to be finite demands that *all* sufficiently large multiples of 6 decompose as sums of two middle numbers, not only those that are themselves middle numbers. Dubner 3 is strictly stronger than MENC: it pins down the exception set exactly. Since Definition 4.9 below gives $\mathcal{X} \supseteq$ A243956 unconditionally, MENC is equivalent to asserting that $\mathcal{X}$ is a finite superset of A243956, and Dubner 3 asserts that superset is in fact equality.

Lemma 4.7 (MENC implies WDC)**.** *Conjecture 4.4 implies Conjecture 4.2.*

Proof. If $\mathcal{X}$ is finite, every sufficiently large positive integer lies in $\mathcal{W} + \mathcal{W}$. In particular, every sufficiently large $w \in \mathcal{W}$ decomposes as a sum of two elements of $\mathcal{W}$. Dubner's computation [3] confirms the finitely many small cases. □

Lemma 4.8 (WDC implies WDCZ)**.** *Conjecture 4.2 implies WDCZ (Conjecture 11.3 in Part 4).*

Proof. WDC asserts that every $w \in \mathcal{W}$ with $w > 1$ decomposes; the set of non-decomposing witnesses is at most $\{1\}$, which has density zero in $\mathcal{W}$. □

Restatement in our terminology.

The conjectures of this section, ordered from strongest to weakest, are:

$$\text{Dubner 3} \Rightarrow \text{MENC} \Rightarrow \text{MNC} \Leftrightarrow \text{WDC} \Rightarrow \text{WDCZ}.$$

MNC and WDC are equivalent (dividing middle numbers by 6); we use WDC throughout the remainder of this paper.

In our notation:

- **Dubner 3** (Conjecture 4.5): $\mathcal{X}$ equals A243956 exactly.
- **MENC** (Conjecture 4.4): $\mathcal{X}$ is finite — equivalently, a finite superset of A243956 (since $\mathcal{X} \supseteq$ A243956 unconditionally by Definition 4.9).
- **MNC** (Conjecture 4.1) $\Leftrightarrow$ **WDC** (Conjecture 4.2): every middle number $6w > 6$ is the sum of two middle numbers; equivalently, every witness $w > 1$ is the sum of two witnesses. Dubner verified this up to 2×10^{10}.
- **WDCZ**: all but a density-zero subset of $\mathcal{W}$ decompose. Defined formally in Part 4.

Definition 4.9 (Exception set)**.**

$$\mathcal{X} = \mathbb{N}_{\geq 1} \setminus \{u + v : u, v \in \mathcal{W}\}.$$

Since $\min \mathcal{W} = 1$, every element of the sumset is at least 2, so $1 \in \mathcal{X}$ automatically despite $1 \in \mathcal{W}$. The known value of $\mathcal{X}$ up to 2×10^{10} is A243956 [10]:

$$\mathcal{X} \supseteq \{1, 16, 67, 86, 131, 151, 186, 191, 211, 226, 541, 701\}.$$

Dubner 3 (Conjecture 4.5) asserts this is an equality; MENC (Conjecture 4.4) asserts only that $\mathcal{X}$ is finite.

Remark 4.10. Dubner's computation confirms that every element of $\mathcal{X}$ greater than 1 lies in $\mathbb{N} \setminus \mathcal{W}$: none of the eleven exceptions beyond 1 is itself a twin prime witness. The element 1 lies in $\mathcal{W}$ (since 5 and 7 are prime) but belongs to $\mathcal{X}$ because no two elements of $\mathcal{W}$ sum to 1. Definition 4.9 is robust to future discoveries: a new exception beyond 701 would falsify Dubner 3 and MENC, but would merely enlarge $\mathcal{X}$ in Definition 4.9, leaving the partition of $\mathbb{N}_{\geq 1}$ intact.

5. Selcoe's Interval Conjecture

Conjecture 5.1 (Selcoe's Interval Conjecture (SIC) [12])**.** For every $k \geq 1$ there exists a twin prime witness $w \in \mathcal{W}$ with $5k \leq w \leq 7k$.

Our terminology. SIC asserts that $\mathcal{W}$ intersects every interval of the form $[5k, 7k]$, i.e. the relative gap between consecutive witnesses is bounded by a factor of $7/5 - 1 = 2/5$. In particular SIC implies consecutive witnesses satisfy $w_{n+1} - w_n \leq w_n$, the same gap bound we derive from BC (Lemma 6.8), and SIC implies TPC directly. The two conjectures reach the same gap bound by different mechanisms: BC via the additive sumset structure of $\mathcal{W}$, SIC via a direct interval-covering assertion. Whether BC and SIC are equivalent or independent is an open question.

Part 2. The Bridging Conjecture

6. The Bridging Conjecture and its Consequences

We now introduce the Bridging Conjecture (BC) and derive its immediate consequences.

Conjecture 6.1 (Bridging Conjecture (BC))**.** For every $V \in \mathbb{N}$ there exist $u, v, w \in \mathcal{W}$ with $u \leq v \leq V < w$ and $u + v = w$.

That is, for any threshold V, there is a pair of witnesses $u, v \leq V$ whose sum $w = u+v$ is itself a witness strictly above V. The witnesses u and v act as a *bridge*: they are known to be below V, yet their sum certifies the existence of a new witness beyond V. We call (u, v) a *bridging pair* for V.

Remark 6.2. BC and WDC (Conjecture 4.2) are related but distinct. WDC asserts that every witness $w > 1$ decomposes as a sum of two smaller witnesses: the decomposition is required to stay *within* $\mathcal{W}$. BC asserts that for every V, some pair below V sums to a witness *above* V: the focus is on crossing the threshold, not on covering $\mathcal{W}$. In particular, BC does not require the bridging witness w to decompose further; and WDC does not by itself guarantee that $\mathcal{W}$ is infinite (a finite set can satisfy WDC vacuously once the bridging pairs run out).

Remark 6.3. BC is strictly weaker than MENC (Conjecture 4.4). MENC asserts $\mathcal{X}$ is finite, which implies $\mathcal{W} + \mathcal{W}$ covers all sufficiently large positive integers. BC requires only that $\mathcal{W} + \mathcal{W}$ is unbounded: for every V, some sum $u + v$ with $u, v \leq V$ exceeds V. This is a much weaker demand; in particular BC is consistent with $\mathcal{X}$ being infinite.

Lemma 6.4 (MENC implies BC)**.** *MENC (Conjecture 4.4) implies BC (Conjecture 6.1), assuming TPC.*

Proof. Given V, TPC gives a smallest element $w \in \mathcal{W}$ with $w > V$. MENC asserts $\mathcal{X}$ is finite; let $M = \max(\mathcal{X})$. For V large enough that $w > M$, we have $w \notin \mathcal{X}$, so MENC gives $w = u + v$ with $u, v \in \mathcal{W}$. By minimality of w, both u and v are at most V. □

Lemma 6.5 (WDC and TPC together imply BC)**.** *Conjecture 4.2 and TPC together imply Conjecture 6.1.*

Proof. Given V, TPC gives a smallest element $w \in \mathcal{W}$ with $w > V$. WDC gives $w = u + v$ with $u, v \in \mathcal{W}$ and $u, v < w$. By minimality of w, we have $u, v \leq V$. □

Remark 6.6. Lemma 6.5 will reappear as the key step in Part 3: we will prove it is not merely a lemma but the converse direction of a theorem, once WDC is placed in the role of a hypothesis rather than an intermediate conjecture.

Theorem 6.7 (BC implies TPC)**.** *The Bridging Conjecture implies the Twin Prime Conjecture.*

Proof. For any $V \in \mathbb{N}$, BC gives $w \in \mathcal{W}$ with $w > V$. Since V is arbitrary, $\mathcal{W}$ is unbounded, i.e. infinite. □

Lemma 6.8 (Witness Gap Bound)**.** *Assuming BC, consecutive elements $w_n < w_{n+1}$ of $\mathcal{W}$ satisfy $w_{n+1} - w_n \leq w_n$.*

Proof. Set $V = w_n$. BC gives $u, v \in \mathcal{W}$ with $u \leq v \leq w_n$ and $w = u+v \in \mathcal{W}$ with $w > w_n$, so $w_{n+1} \leq w = u+v \leq 2w_n$, hence $w_{n+1} - w_n \leq w_n$. □

Corollary 6.9 (Lower bound on twin prime density)**.** *Assuming BC,* $|\mathcal{W} \cap [1, x]| \gg \log x$.

Proof. By Lemma 6.8, every interval $(w_n, 2w_n]$ contains an element of $\mathcal{W}$. Applying this inductively from $w_1 = 1$ gives at least $\lfloor \log_2 x \rfloor$ elements up to x. □

Remark 6.10. The bound $|\mathcal{W} \cap [1, x]| \gg \log x$ is far below the Hardy–Littlewood prediction $|\mathcal{W} \cap [1, x]| \sim C_2\, x/(\log x)^2$, but it is a purely additive consequence of BC with no appeal to sieve methods. The BC gap bound $O(w_n)$ is also much weaker than the Cramér-type heuristic ($O((\log w_n)^2)$) for twin prime witnesses; the latter is consistent with Oppermann's Conjecture applied to the twin prime density.

Part 3. The Witness Decomposition Conjecture and its Implications

7. The Logical Landscape

We now turn to the relationship between WDC, BC, and TPC. We have established in Part 2 that BC ⇒ TPC (Theorem 6.7). We have not yet established the converse TPC ⇒ BC, nor the converse direction TPC ⇒ BC under any additional hypothesis.

WDC (Conjecture 4.2) is the natural hypothesis that closes the loop. To see why, consider what TPC alone gives: it guarantees a next witness w above any threshold V, but it does not guarantee that w is reachable as a sum of two smaller witnesses. WDC supplies exactly this: every witness $w > 1$ is the sum of two smaller witnesses. So TPC gives the witness, and WDC gives the decomposition, and together they yield a bridging pair.

Before proving the theorem we survey the full logical landscape. There are $2^3 = 8$ combinations of truth values for WDC, BC, TPC. Theorem 6.7 eliminates two: BC cannot be true while TPC is false. The main theorem of this part (Theorem 8.2) eliminates one more: under WDC, BC and TPC cannot diverge. This leaves exactly five logically consistent worlds.

WDC	BC	TPC	Consistent?	Notes
T	T	T	✓	World 1: conjectured actual world
T	T	F	×	Eliminated by Thm 6.7
T	F	T	×	Eliminated by Thm 8.2
T	F	F	✓	World 4
F	T	T	✓	World 5
F	T	F	×	Eliminated by Thm 6.7
F	F	T	✓	World 7
F	F	F	✓	World 8

We characterise each of the five consistent worlds in Section 9. First we prove the theorem.

8. WDC implies the equivalence of BC and TPC

We state the result first as a conjecture, since the reader may wish to pause and verify the logical structure before seeing the proof.

Conjecture 8.1 (Candidate theorem)**.** WDC $\Rightarrow$ (BC $\Leftrightarrow$ TPC).

We verify the conjecture by examining each direction of the biconditional under the hypothesis WDC.

Direction 1: BC $\Rightarrow$ TPC (unconditional). This is Theorem 6.7; WDC is not needed.

Direction 2: TPC $\Rightarrow$ BC (assuming WDC). Suppose $\mathcal{W}$ is infinite (TPC) and every $w \in \mathcal{W}$ with $w > 1$ decomposes as a sum of two smaller elements of $\mathcal{W}$ (WDC). Fix any $V \in \mathbb{N}$. TPC gives a smallest element $w \in \mathcal{W}$ with $w > V$. WDC (applied to $w > 1$) gives $u, v \in \mathcal{W}$ with $u, v < w$ and $u + v = w$. By minimality of w, every element of $\mathcal{W}$ below w is at most V, so $u, v \leq V$. Thus (u, v, w) is a bridging triple for V, and BC holds at V. Since V was arbitrary, BC holds. □

We have established both directions. Conjecture 8.1 is therefore a theorem.

Theorem 8.2 (WDC implies BC $\Leftrightarrow$ TPC)**.** *$WDC \Rightarrow (BC \Leftrightarrow TPC)$.*

Proof. The direction BC $\Rightarrow$ TPC is Theorem 6.7 and holds unconditionally. The direction TPC $\Rightarrow$ BC under WDC is established in Direction 2 above. □

Remark 8.3. The converse WDC $\Leftarrow$ BC does not hold in general. BC asserts that for every V some pair of witnesses below V sums to a witness above V; it says nothing about whether the witnesses above V themselves decompose as sums of two smaller witnesses. In World 5 (WDC false, BC true, TPC true), BC holds but some witnesses lack any decomposition at all. Thus WDC is strictly stronger than BC.

Remark 8.4. The role of WDC in Theorem 8.2 is precisely to supply the decomposition of the smallest witness above V. Without WDC, TPC alone guarantees a witness $w > V$ exists, but not that w is reachable as a sum of two witnesses below V. The gap between TPC and BC is exactly the gap WDC fills.

9. The Five Possible Worlds

Theorems 6.7 and 8.2 together reduce the eight combinations of truth values to five. We characterise each world below in terms of the structure of $\mathcal{W}$ and the exception set $\mathcal{X}$.

World 1: WDC true, BC true, TPC true. The conjectured actual world. $\mathcal{W}$ is infinite; every V has a bridging pair; every witness $w > 1$ decomposes as a sum of two smaller witnesses. MENC predicts $\mathcal{X}$ is finite and equal to A243956.

World 4: WDC true, BC false, TPC false. $\mathcal{W}$ is finite with largest element w^*. WDC holds for all elements of $\mathcal{W}$ (vacuously beyond w^*). $\mathcal{X}$ is co-finite: all sufficiently large positive integers are exceptions.

World 5: WDC false, BC true, TPC true. $\mathcal{W}$ is infinite and every V has a bridging pair (BC), but some witness $w > 1$ has no decomposition as a sum of two smaller witnesses (WDC fails). $\mathcal{X}$ may be infinite. BC holds via a *relay* mechanism: the decomposable witnesses collectively bridge every threshold V, compensating for the gaps left by witnesses that have no progenitor pair within $\mathcal{W}$.

World 7: WDC false, BC false, TPC true. $\mathcal{W}$ is infinite (TPC holds), but some threshold V^* has no bridging pair (BC fails at V^*). WDC also fails. The witnesses beyond V^* exist but cannot be reached as sums of witnesses below V^*: the additive structure of $\mathcal{W}$ stalls at V^* even though $\mathcal{W}$ itself continues.

World 8: WDC false, BC false, TPC false. $\mathcal{W}$ is finite. Both BC and WDC fail. $\mathcal{X}$ is co-finite.

10. Discussion

What WDC adds. Theorem 8.2 shows that WDC is the exact hypothesis needed to promote TPC to BC. By itself, TPC guarantees the existence of infinitely many witnesses but says nothing about their additive structure. WDC is a structural claim: it asserts that the additive self-reinforcement of $\mathcal{W}$ is complete, in the sense that every witness participates in the sumset. Under WDC, knowing that $\mathcal{W}$ is infinite is sufficient to know that $\mathcal{W}$ is additively unbounded (BC), and conversely.

WDC as a flank attack on TPC. An alternative strategy for approaching TPC is to prove WDC and BC separately. WDC is a combinatorial statement about multiples of 6: it asserts that the sumset $\mathcal{W} + \mathcal{W}$ covers $\mathcal{W} \setminus \{1\}$. BC is an additive bootstrapping principle: if $\mathcal{W}$ is non-empty up to V, it certifies a further element beyond V. Together, by Theorem 8.2, they yield TPC. Approaches to WDC may be amenable to additive number theory tools (circle method, density arguments) that are blocked by the parity barrier in direct sieve approaches to TPC; this is the sense in which WDC offers a "flank attack" on the problem.

BC is strictly weaker than WDC. Remark 8.3 records that BC does not imply WDC. This matters for the programme above: proving BC alone does not complete the argument; WDC must be established independently.

Relation to other gap conjectures. The gap bound $w_{n+1} - w_n \leq w_n$ (Lemma 6.8) places BC in a broader family of conjectures asserting that gaps grow no faster than the sequence itself. Oppermann's Conjecture (Section 2) asserts the analogous $O(\sqrt{p_k})$ bound for ordinary primes; SIC (Section 5) reaches the same $O(w_n)$ bound for witnesses by a direct interval-covering argument. BC, Oppermann, and SIC are mutually independent as stated; they share a structural character but concern different sequences and different scales.

Computational status. BC has been verified computationally for all V up to 2×10^{10} as a consequence of Dubner's verification of WDC [3]. Heuristic and computational evidence for BC beyond this range, including gap ratio plots and decomposition count analysis to $w = 400{,}000$, is developed in [1].

Part 4. Density-Zero Variants

Parts 2 and 3 establish the hard logical core: BC $\Rightarrow$ TPC unconditionally, and WDC $\Rightarrow$ (BC $\Leftrightarrow$ TPC). Here we introduce *density-zero variants* of BC, WDC, and MENC. These weaken each conjecture by allowing a density-zero set of exceptions, and we show the same implication framework survives.

11. Definitions of the Z-Variants

Definition 11.1 (Density-zero set)**.** A set $S \subseteq \mathbb{N}$ has *density zero* if $|S \cap [1, x]|/x \to 0$ as $x \to \infty$.

Conjecture 11.2 (BCZ: Bridging Conjecture up to density zero)**.** There exists a density-zero set $E \subseteq \mathbb{N}$ such that for every $V \in \mathbb{N} \setminus E$ there exist $u, v, w \in \mathcal{W}$ with $u \leq v \leq V < w$ and $u + v = w$.

That is, BC fails for at most a density-zero set of thresholds.

Conjecture 11.3 (WDCZ: Witness Decomposition Conjecture up to density zero)**.** Let $F = \{w \in \mathcal{W} : w \notin \mathcal{W} + \mathcal{W}\}$ be the set of non-decomposing witnesses, listed in increasing order as $w_1 < w_2 < \cdots$, and set $w_0 = 0$. Then

$$\sum_{\substack{w_n \in F \\ w_n \leq x}} (w_n - w_{n-1}) = o(x).$$

That is, the intervals before non-decomposing witnesses occupy a vanishing fraction of $[1, x]$.

Remark 11.4. WDCZ is a gap-weighted density condition on F. It is implied by — but strictly stronger than — the counting condition $|F \cap [1, x]| = o(x)$, unless the gaps $w_n - w_{n-1}$ before non-decomposing witnesses are uniformly bounded. In particular, if BC holds then the gap bound $w_n - w_{n-1} \leq w_{n-1}$ gives $\sum_{w_n \in F, w_n \leq x} (w_n - w_{n-1}) \leq \sum_{w_n \in F, w_n \leq x} w_n$, and the counting condition then implies WDCZ under Hardy–Littlewood asymptotics. WDCZ is the minimal condition that makes the proof of Theorem 12.4 work directly.

Conjecture 11.5 (MENCZ: Middle Even Number Conjecture up to density zero)**.** The exception set $\mathcal{X} = \mathbb{N}_{\geq 1} \setminus (\mathcal{W} + \mathcal{W})$ has density zero in $\mathbb{N}$.

12. Implications among the Z-variants

The full implication lattice, combining the hard results with the Z-variants, is:

$$\begin{array}{ccccc} \text{Dubner 3} & \Rightarrow & \text{MENC} & \Rightarrow & \text{WDC} \\ & & \Downarrow & & \Downarrow \\ & & \text{MENCZ} & \xrightarrow{\text{counting only}} & \text{WDCZ} \end{array}$$

The leftward implication MENC $\Rightarrow$ WDC is Lemma 4.7; WDC $\Rightarrow$ WDCZ is Lemma 4.8. We now establish the remaining implication MENCZ $\Rightarrow$ WDCZ, and then the BCZ side.

Lemma 12.1 (MENCZ implies counting-density of F)**.** *Conjecture 11.5 implies that F has counting density zero: $|F \cap [1, x]| = o(x)$.*

Proof. $F = \mathcal{W} \setminus (\mathcal{W} + \mathcal{W}) \subseteq \mathcal{X}$, and $\mathcal{X}$ has density zero by MENCZ, so F is a subset of a density-zero set and hence has density zero. □

Remark 12.2. The counting condition $|F \cap [1, x]| = o(x)$ is weaker than the gap-weighted WDCZ (Conjecture 11.3); see Remark 11.4. MENCZ therefore does not directly imply gap-weighted WDCZ without an additional bound relating the gaps $w_n - w_{n-1}$ before non-decomposing witnesses to the density of F. The implication arrow MENCZ $\Rightarrow$ WDCZ in the diagram above should be read as implying the counting-density form of WDCZ. The gap-weighted form is the hypothesis actually used in the proof of Theorem 12.4, and is the content of Conjecture 11.3 as stated.

Lemma 12.3 (BCZ implies TPC)**.** *Conjecture 11.2 implies TPC.*

Proof. Let E be the density-zero exception set for BCZ. Since E has density zero, $\mathbb{N} \setminus E$ is infinite. For any $V \in \mathbb{N} \setminus E$, BCZ gives a witness $w > V$. Hence $\mathcal{W}$ is unbounded, i.e. infinite. □

Theorem 12.4 (WDCZ implies BCZ $\Leftrightarrow$ TPC)**.** *WDCZ $\Rightarrow$ (BCZ $\Leftrightarrow$ TPC).*

Proof. The direction BCZ $\Rightarrow$ TPC is Lemma 12.3 and holds unconditionally.

For TPC $\Rightarrow$ BCZ under WDCZ: assume $\mathcal{W}$ is infinite and let $F = \{w \in \mathcal{W} : w \notin \mathcal{W} + \mathcal{W}\}$ satisfy the WDCZ condition. BCZ fails at V iff the smallest $w \in \mathcal{W}$ above V lies in F: for if $w \notin F$ then $w = u + v$ with $u, v \in \mathcal{W}$, and by minimality of w both $u, v \le V$, giving a bridging pair. Thus the BCZ failure set satisfies

$$E \subseteq \bigcup_{w_n \in F} [w_{n-1}, w_n),$$

where these intervals are disjoint, so

$$|E \cap [1,x]| \;\le\; \sum_{\substack{w_n \in F \\ w_n \le x}} (w_n - w_{n-1}) \;=\; o(x)$$

by WDCZ. Hence E has density zero and BCZ holds. □

Remark 12.5. The two conjectures play asymmetric roles. BCZ alone implies TPC (Lemma 12.3): if bridging succeeds for a density-one set of thresholds then $\mathcal{W}$ is infinite. WDCZ alone does not imply TPC: if $\mathcal{W}$ is finite then every non-decomposing witness contributes a finite gap, the weighted sum is finite and hence $o(x)$, so WDCZ holds vacuously. What WDCZ supplies is the missing direction TPC ⇒ BCZ: knowing $\mathcal{W}$ is infinite is not enough to guarantee bridging almost everywhere, but knowing almost all witnesses decompose is exactly what is needed to convert each new witness above V into a bridging pair below V. The parallel with the hard case is exact: BCZ alone suffices for TPC, just as BC alone suffices for TPC; and WDCZ is to BCZ what WDC is to BC — the structural hypothesis that closes the equivalence in the other direction.

A further subtlety: if WDC is false but WDCZ is true, then BC itself may fail — there will be thresholds V just below a non-decomposing witness where no bridging pair exists. But provided those witnesses are sparse in the gap-weighted sense, BCZ still holds: the density-one set of decomposing witnesses collectively bridges almost every threshold, compensating for the gaps that BC cannot cross. This is the density-zero analogue of World 5: BC fails, but BCZ survives.

13. Discussion of Z-variants

Why the Z-variants matter. Dubner 3 is the strongest statement: it pins $\mathcal{X}$ to a specific named set. MENC weakens this to finiteness of $\mathcal{X}$, without specifying the set. WDC weakens further to the witnesses only: every witness decomposes. WDCZ weakens WDC to almost every witness. BCZ is weaker still: bridging pairs exist for almost all thresholds.

The unconditional results are:

$$\text{BC} \Rightarrow \text{TPC} \qquad \text{and} \qquad \text{BCZ} \Rightarrow \text{TPC}.$$

WDC and WDCZ each close the loop in their respective settings (Theorems 8.2 and 12.4):

$$\text{WDC} \Rightarrow (\text{BC} \Leftrightarrow \text{TPC}),$$
$$\text{WDCZ} \Rightarrow (\text{BCZ} \Leftrightarrow \text{TPC}).$$

The strengthening chain from Dubner 3 down to WDCZ supplies progressively weaker sufficient conditions for WDC (and hence for closing the loop):

$$\text{Dubner 3} \Rightarrow \text{MENC} \Rightarrow \text{WDC} \Rightarrow \text{WDCZ}.$$

The minimal hypothesis for TPC in this framework is BCZ alone (unconditionally), or WDCZ together with TPC to get BC.

Relation to the hard results. The hard theorems in Parts 2 and 3 are unconditional: BC $\Rightarrow$ TPC holds for every V, not almost every V. The Z-variants suggest a programme: prove the weaker BCZ first (perhaps via additive number theory or ergodic arguments), then tighten to BC.

BCZ as a sufficient and potentially easier route to TPC. Lemma 12.3 shows that BCZ alone implies TPC — no appeal to WDCZ, WDC, or any other conjecture is needed. This means that proving BCZ is a complete strategy for proving TPC.

BCZ may be substantially easier to prove than BC. BC requires a bridging pair at *every* threshold V; BCZ only requires one at almost every threshold. Sieve methods and density arguments typically yield results that hold almost everywhere — they are well-suited to establishing density-one conclusions — whereas proving a statement for every V without exception is often out of reach. The parity barrier and other obstructions that block direct sieve approaches to TPC may be navigable in the density-zero setting where a sparse set of failures is permitted.

Separately, Theorem 12.4 shows that WDCZ and TPC are equivalent (modulo BCZ): if one can prove WDCZ and TPC independently, BCZ follows for free. But the primary strategic point is simpler: *to prove TPC it suffices to prove BCZ*, and BCZ asks only that bridging fails on a density-zero set of thresholds — a weaker target than BC.

As Euclid opined, there is no royal road to geometry; but with Turing's machines there might well be a railway.

Relation to Goldbach. MENCZ is a close analogue of the even Goldbach conjecture restricted to multiples of 6, allowing finitely many exceptions. The even Goldbach conjecture has been verified to 4×10^{18} and a density-zero version (every even number is the sum of two primes except for a density-zero set) follows from standard sieve estimates. Whether analogous sieve estimates apply to $\mathcal{W} + \mathcal{W}$ is an open question, but the structural parallel is encouraging.

Acknowledgements

AI systems (Claude, Aristotle, ChatGPT, Gemini) contributed to the development of this paper. The nature and extent of that collaboration is documented in full in the companion paper [2]. All mathematical ideas, conjectures, and conclusions are the author's own. The author takes full responsibility for the content of this paper.

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