Exact 2-adic Valuations of $3n + 1$ and the Geometric Mean Contraction Rate of the Syracuse Map

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July 2026

Abstract

We prove exact results on the 2-adic valuation $v_2(3n+1)$ under the Syracuse map $S(n) = (3n+1)/2^{v_2(3n+1)}$, classified by the residue of n modulo 8. For $n \equiv 1, 3, 7 \pmod 8$ the valuation is a *pointwise constant*: $v_2(3n+1) = 2, 1, 1$ respectively, for every n in the class. For $n \equiv 5 \pmod 8$, the valuation varies but has *exact expectation* 4 under natural density: writing $n = 8k + 5$ gives $v_2(3n + 1) = 3 + v_2(3k + 2)$, and $E[v_2(3k+2)] = 1$ follows from $\gcd(3, 2^j) = 1$, which makes $3k+2$ uniformly distributed modulo 2^j for every $j \geq 1$.

Writing $c_r = E[v_2(3n + 1) \mid n \equiv r \pmod 8]$, the four expected valuations sum to $c_1 + c_3 + c_5 + c_7 = 2 + 1 + 4 + 1 = 8$, so their geometric mean satisfies $(3^4/2^8)^{1/4} = 3/4$. This is the geometric mean asymptotic contraction rate of the Syracuse map across all four letter types, and is the arithmetic origin of the 3/4 decay appearing in the Steiner sentence length distribution $P(\text{length} = k) = 3^{k-1}/4^k$ proved in the companion paper [3].

We also prove the full distribution of $v_2(3n+1)$ for $n \equiv 5 \pmod 8$: $P(v_2(3n+1) = j \mid n \equiv 5 \pmod 8) = 1/2^{j-2}$ for $j \geq 3$. All results are proved unconditionally (no appeal to the Collatz conjecture) and have been machine-verified in Lean 4/Mathlib with no `sorry`.

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1 Background and Notation

We work with odd positive integers throughout. The *2-adic valuation* $v_2(n)$ denotes the exponent of the largest power of 2 dividing n.

Definition 1.1 (Syracuse map)**.** The *Syracuse map* on odd positive integers is

$$S(n) \;=\; \frac{3n+1}{2^{v_2(3n+1)}}, \qquad n \text{ an odd positive integer.}$$

The ratio $S(n)/n = (3n+1)/(n\cdot 2^{v_2(3n+1)})$ measures the multiplicative change per Syracuse step. Its asymptotic behaviour as $n \to \infty$ is governed entirely by the exponent $v_2(3n+1)$, since $(3n+1)/n \to 3$. The central question of this paper is: what is $v_2(3n+1)$, and how does it depend on $n \bmod 8$?

A *Steiner sentence* of length k is a string of k consecutive Steiner words (maximal path segments whose compressed mod-8 form matches (7*3)?(1 | 5)) in which the first $k-1$ words terminate at letter 1 and the final word terminates at letter 5. The sentence length distribution $P(\text{length} = k) = 3^{k-1}/4^k$ is proved in the companion paper [3]; the valuation results in the present paper supply the arithmetic foundation for that proof. The Steiner language hierarchy — alphabet, letters, words, and sentences — is defined in Appendix A of Paper 67 [3].

2 Main Theorem: Valuations by Residue Class

Theorem 2.1 (2-adic valuations of $3n+1$ mod 8)**.** *Let n be an odd positive integer and write $c_r = E[v_2(3n+1) \mid n \equiv r \pmod 8]$ where the expectation is over the uniform (natural) density on the residue class. Then:*

1. $n \equiv 1 \pmod 8$*:* $v_2(3n+1) = 2$ *for every such n, so* $c_1 = 2$.
2. $n \equiv 3 \pmod 8$*:* $v_2(3n+1) = 1$ *for every such n, so* $c_3 = 1$.
3. $n \equiv 5 \pmod 8$*:* $v_2(3n+1) \geq 3$ *for every such n, and* $E[v_2(3n+1) \mid n \equiv 5 \pmod 8] = 4$*, so* $c_5 = 4$.
4. $n \equiv 7 \pmod 8$*:* $v_2(3n+1) = 1$ *for every such n, so* $c_7 = 1$.

Proof. Write $n = 8k + r$ for $r \in \{1, 3, 5, 7\}$ and $k \geq 0$.

(1) $r = 1$**:** $3n + 1 = 3(8k + 1) + 1 = 24k + 4 = 4(6k + 1)$. Since $6k$ is even, $6k + 1$ is odd, so $v_2(6k + 1) = 0$ and $v_2(3n + 1) = v_2(4) + v_2(6k + 1) = 2 + 0 = 2$ for every k.

(2) $r = 3$**:** $3n + 1 = 3(8k + 3) + 1 = 24k + 10 = 2(12k + 5)$. Since $12k$ is even, $12k + 5$ is odd, so $v_2(3n + 1) = 1$ for every k.

(3) $r = 5$**:** $3n+1 = 3(8k+5)+1 = 24k+16 = 8(3k+2)$. Hence $v_2(3n+1) = 3+v_2(3k+2)$ for every k, establishing $v_2(3n + 1) \geq 3$. It remains to show $E[v_2(3k + 2)] = 1$.

For any $j \geq 1$, since $\gcd(3, 2^j) = 1$, multiplication by 3 is a bijection on $\mathbb{Z}/2^j\mathbb{Z}$. Therefore as k ranges uniformly over $\mathbb{Z}/2^j\mathbb{Z}$, the value $3k + 2 \pmod{2^j}$ is also uniform over $\mathbb{Z}/2^j\mathbb{Z}$. In particular:

$$P\big(2^j \mid 3k + 2\big) = \frac{1}{2^j}.$$

Using the standard identity for non-negative integer-valued random variables:

$$E[v_2(3k + 2)] = \sum_{j=1}^{\infty} P(v_2(3k + 2) \geq j) = \sum_{j=1}^{\infty} P\big(2^j \mid 3k + 2\big) = \sum_{j=1}^{\infty} \frac{1}{2^j} = 1.$$

Therefore $c_5 = E[v_2(3n + 1) \mid n \equiv 5 \pmod 8] = 3 + 1 = 4$.

(4) $r = 7$**:** $3n + 1 = 3(8k + 7) + 1 = 24k + 22 = 2(12k + 11)$. Since $12k + 11$ is odd ($12k$ even, $+11$ odd), $v_2(3n + 1) = 1$ for every k. □

Remark 2.2. Parts (1), (2), (4) are exact pointwise statements: the valuation is a fixed integer on the entire residue class. Part (3) is an expectation statement: the valuation varies over the class $n \equiv 5 \pmod 8$, but its mean is exactly 4. The key tool in part (3) is the invertibility of multiplication by 3 modulo 2^j, i.e. $\gcd(3, 2^j) = 1$, which makes the distribution of $3k + 2 \pmod{2^j}$ exactly uniform.

Remark 2.3. Theorem 2.1 explains the mod-8 structure of the Syracuse map: for $n \equiv 1 \pmod 8$, every step removes exactly two factors of 2; for $n \equiv 3$ or $7 \pmod 8$, exactly one factor of 2 is removed; for $n \equiv 5 \pmod 8$, at least three factors of 2 are removed (exactly three when $3k + 2$ is odd, more otherwise). The detailed step taxonomy and its connection to Collatz path structure are developed in [1].

3 The Balance Identity

Corollary 3.1 (Balance of expected valuations)**.** *The expected valuations satisfy:*

$$c_1 + c_3 + c_5 + c_7 = 2 + 1 + 4 + 1 = 8 = 4 \times 2.$$

The mean expected valuation across all four classes is exactly $2 = c_1$*, the constant valuation of the neutral class* $1 \pmod 8$*.*

Equivalently, in terms of powers of 2 *and* 3*:*

$$\frac{3^4}{2^{c_1} \cdot 2^{c_3} \cdot 2^{c_5} \cdot 2^{c_7}} = \frac{3^4}{2^8} = \frac{81}{256} = \left(\frac{3}{4}\right)^4.$$

Proof. Direct: $2 + 1 + 4 + 1 = 8$ and $2^{c_1} \cdot 2^{c_3} \cdot 2^{c_5} \cdot 2^{c_7} = 2^{2+1+4+1} = 2^8 = 256$. □

Remark 3.2. The balance identity has a clean interpretation in terms of the asymptotic contraction ratios. For large n in class r, the Syracuse step multiplies n by approximately $3/2^{c_r}$:

$$\frac{S(n)}{n} = \frac{3n+1}{n \cdot 2^{v_2(3n+1)}} \longrightarrow \frac{3}{2^{c_r}} \quad \text{as } n \to \infty.$$

The four asymptotic ratios are 3/4, 3/2, 3/16, 3/2 for classes $1, 3, 5, 7$ respectively. Their product is $3^4/2^8 = (3/4)^4$, whose fourth root is 3/4. That is: the *geometric mean* of the four asymptotic contraction ratios equals the ratio of the neutral class 1.

Note that the ratios $3/2^{c_r}$ are exact as limits ($n \to \infty$) for classes 1, 3, 7 where v_2 is constant, and exact as expectations for class 5. The finite-n correction is $O(1/n)$ in each case and does not affect the integer-exponent result.

Remark 3.3 (Asymmetry of the four classes). The four classes play structurally distinct roles:

- Classes 3 and 7 are *growth* classes: $c_3 = c_7 = 1$, asymptotic ratio $3/2 > 1$.
- Class 1 is the *neutral* class: $c_1 = 2$, asymptotic ratio $3/4 < 1$.
- Class 5 is the *contraction* class: $c_5 = 4$, asymptotic ratio $3/16 \ll 1$. A single step in class 5 contributes $c_5 = 4$ to the sum of valuations, the same as *four* steps in classes 3 or 7, or *two* steps in class 1.

The balance $c_1 + c_3 + c_5 + c_7 = 4c_1$ is equivalent to saying class 5's excess valuation $(c_5 - c_1) = 2$ exactly compensates the deficit of classes 3 and 7 combined: $(c_1 - c_3) + (c_1 - c_7) = 2$.

4 Dynamical Implications

4.1 Class 5 as a Dynamical Sink

Theorem 2.1 gives a precise, quantitative sense in which $5 \pmod 8$ acts as a *dynamical sink* in the Collatz tree. Each visit to a $5 \pmod 8$ node contributes an expected valuation of 4 to the denominator of the Syracuse ratio, compared to 2 for class 1, and only 1 for classes 3 and 7. In the asymptotic ratio sense, class 5 contracts by $3/2^4 = 3/16$ per step — four times the exponent of class 1 ($3/2^2 = 3/4$) and sixteen times the inverse of classes 3 and 7 ($3/2^1 = 3/2$).

4.2 Connection to 5-Dominance in Collatz Paths

It is empirically observed that the vast majority of Collatz orbits pass through the integer 5 on the way to 1. Theorem 2.1 gives the structural foundation: any orbit entering the 5 (mod 8) class experiences a valuation of 4 in expectation, driving it toward 1 far more rapidly than any other class. The small complementary fraction $\approx 1/16$ corresponds to the mod-32 edge density of $5 \to 5$ transitions in the Syracuse graph (see Paper 65 [2]).

4.3 Connection to Steiner Sentence Length Distribution

Paper 65 [2] empirically observes that Steiner sentence lengths follow $P(k) = \frac{1}{3}(3/4)^k$ rather than the naive $(1/2)^k$. The balance identity $c_1 + c_3 + c_5 + c_7 = 4c_1$ is the exact integer expression of the 3/4 decay rate: the neutral class 1 (which governs within-sentence dynamics after conditioning on non-termination) has valuation exactly $c_1 = 2$, giving asymptotic ratio 3/4. The first-principles proof is given in Paper 67 [3].

A key insight of Paper 67 is that Steiner sentences are the *right scale* for this analysis precisely because they make class 5 a sink rather than a source. Within a sentence, the chain visits only classes 1, 3, and 7 until it terminates at class 5. For these three classes, the 2-adic valuation $v_2(3n + 1)$ is a *constant* (2, 1, 1 respectively — Theorem 2.1), so the transition probabilities are exact rationals with no averaging required. The variable-valuation behaviour of class 5 — the central subject of this paper, requiring the geometric series argument of Theorem 2.1(3) — never appears in the interior of a sentence. It enters only at the sentence boundary (the uniform output of class 5 that starts each new sentence) and as the termination condition (class 5 is the absorbing state). In both roles, only row 5 of the transition matrix matters, and that row is uniform by the same $\gcd(3, 2^j) = 1$ argument. The two papers are thus complementary: this paper characterises the hard case (class 5, unbounded valuation); Paper 67 sidesteps it by choosing the right scale.

The Steiner language hierarchy — alphabet, letters, words, and sentences — is defined in full in Appendix A of Paper 67 [3].

5 Further Results

Proposition 5.1 (Full distribution for class 5)**.** *For* $n \equiv 5 \pmod 8$ *drawn with natural density,*

$$P\big(v_2(3n+1) = j \bigm| n \equiv 5 \pmod 8\big) \;=\; \frac{1}{2^{j-2}}, \qquad j \geq 3.$$

Proof. Writing $n = 8k + 5$ gives $v_2(3n + 1) = 3 + v_2(3k + 2)$ (as in Theorem 2.1(3)). The pmf of $v_2(3k + 2)$ under natural density is $P(v_2(3k + 2) = m) = 1/2^{m+1}$ for $m \geq 0$: this follows from $P(v_2(3k + 2) \geq m) = P(2^m \mid 3k + 2) = 1/2^m$ (by the uniformity argument of Theorem 2.1(3)) via $P(v_2 = m) = P(v_2 \geq m) - P(v_2 \geq m + 1) = 1/2^m - 1/2^{m+1} = 1/2^{m+1}$. Setting $j = 3 + m$ gives $P(v_2(3n + 1) = j) = 1/2^{(j-3)+1} = 1/2^{j-2}$. Normalisation: $\sum_{j\geq 3} 1/2^{j-2} = \sum_{i\geq 1} 1/2^i = 1$. □

Remark 5.2. This proposition was stated as Open Problem 2 in a draft of this paper. It was machine-verified in Lean 4/Mathlib by the Aristotle automated proof assistant (July 2026); see `ARISTOTLE-REVIEW-66.md` and `Paper66.lean` in the paper directory.

Open problems

1. **Higher moduli.** The argument for class 5 uses $\gcd(3, 2^j) = 1$ to establish uniform distribution of $3k + 2 \pmod{2^j}$. Do analogous exact valuation results hold for finer residue classes modulo 16, 32, and beyond? The mod-32 structure of $5 \to 5$ edges in the Syracuse graph suggests a rich mod-32 refinement.

The bridge from Theorem 2.1 to a proof of $P(\text{sentence length} = k) = 3^{k-1}/4^k$ is given in the companion Paper 67 [3].

Acknowledgements

This paper forms part of a programme of incremental research on the Collatz conjecture. Paper 33 [1] (in preparation) provides the Steiner circuit framework. Paper 65 [2] provides the empirical motivation. All main results have been machine-verified in Lean 4/Mathlib; the Lean source file `Paper66.lean` is included in the arXiv submission. Proof structure and exposition were developed with AI assistance (Claude, Anthropic; Gemini, Google; Aristotle, Harmonic); all mathematical content has been independently verified.

References

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