Rigidity of the Syracuse Transition Matrix: Almost-Everywhere Exact Residue-to-Residue Transitions and the 2-adic Singularity $-1/3$

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Abstract

We study the residue-to-residue transition matrix $T[M]$ of the Syracuse (odd-Collatz) map $S(n) = (3n+1)/2^{v_2(3n+1)}$, where row r and column j record the probability that $S(n) \equiv j \pmod{M}$ given $n \equiv r \pmod{M}$. We prove that for each odd residue r, setting $v = v_2(3r+1)$, the row r of $T[M]$ is *exactly determined* — independent of any probability measure or corpus — whenever $2^{v+1} \mid M$. In that case the targets form a single arithmetic progression of length 2^v hit with equal probability $1/2^v$, and all other entries are exactly zero. The only corpus-dependent rows at modulus M are those r for which $2^{v+1} \nmid M$, i.e. those where $v_2(3r+1)$ is too large to be pinned down by $n \bmod M$ alone. We show that at each power of two $M = 2^k$, the exceptional residue class is represented by $r_k = (4^k - 1)/3$ (satisfying $3r_k + 1 = 4^k$), and that the sequence $r_1, r_2, r_3, \ldots = 1, 5, 21, 85, \ldots$ converges 2-adically to $-1/3$. Thus *all non-rigidity of the Syracuse transition matrix, at every modulus, accumulates at the 2-adic singularity* $-1/3$. We confirm all theoretical predictions with chi-squared tests on five independent corpora of up to 65,536 Steiner paragraphs.

Contents

1 Introduction

The Syracuse map (also called the odd Collatz map) is defined on positive odd integers by

$$S(n) \;=\; \frac{3n+1}{2^{v_2(3n+1)}},$$

where $v_2(m)$ denotes the 2-adic valuation of m. The Collatz conjecture asserts that every positive integer eventually reaches 1 under iteration of the related map $n \mapsto 3n+1$ (if odd) or $n/2$ (if even); the Syracuse formulation collapses the even steps so that each application of S moves directly from one odd number to the next.

A natural object of study is the *residue-to-residue transition matrix* $T[M]$: the $M \times M$ matrix whose (r, j) entry records the probability that $S(n) \equiv j \pmod{M}$ given that $n \equiv r \pmod{M}$, where the probability is taken over a "typical" distribution of odd n in residue class r. Such matrices arise naturally in the probabilistic and ergodic study of the Collatz problem [2].

The 2-adic structure of the Collatz map and its parity distributions have been studied extensively, tracking back to the structural periodicity theorems of Terras [5] and the 2-adic measure-space constructions of Lagarias [1]. Terras showed that parity vectors of length k are periodic with period 2^k and cover all binary strings uniformly; Lagarias showed that the parity map induces a measure-preserving isomorphism on $\mathbb{Z}_2$. These results concern k steps of the standard Collatz map under the modulus 2^k.

The contribution of the present paper is to translate these structural insights into a *definitive single-step characterisation of the Syracuse residue transition matrix $T[M]$ for an arbitrary modulus M*. We prove that what is often treated as a statistical or measure-theoretic distribution is, for almost all residue rows, an *exact arithmetic identity*: the transition probability is exactly $1/2^v$ or 0, determined entirely by $v_2(3r+1)$ with no limiting process or probabilistic assumption required. Furthermore, we explicitly identify the unique sequence of non-rigid residues as 2-adic approximations converging to the pole $-1/3 \in \mathbb{Z}_2$, the point at which the 2-adic valuation of $3n+1$ blows up.

Organisation. Section 2 establishes notation and defines the transition matrix precisely. Section 3 proves the main rigidity theorem. Section 4 characterises the exceptional sequence and its 2-adic limit. Section 5 presents empirical verification across five corpora and fourteen moduli. Section 6 discusses implications.

2 Setup and Definitions

Definition 1. *Since the Syracuse map sends odd integers to odd integers, both source and target residues are restricted to the set of odd residues mod M. For odd residues*

$r, j \in \{1, 3, 5, \ldots\} \cap [0, M)$, *define the* transition matrix entry

$$T[M]_{r,j} \;=\; \lim_{N\to\infty} \frac{\#\{n \le N : n \text{ odd},\ n \equiv r \pmod M,\ S(n) \equiv j \pmod M\}}{\#\{n \le N : n \text{ odd},\ n \equiv r \pmod M\}},$$

when this limit exists and is independent of the sequence of N. The matrix $T[M]$ is thus $\phi(M)/2 \times \phi(M)/2$ (for even M) rather than $M \times M$; even residues do not appear.

In practice we work with finite corpora (defined below) and show that the theoretical predictions are confirmed to high precision.

Definition 2 (Steiner paragraph)**.** *A* Steiner paragraph *rooted at a positive odd integer ℓ (the* leaf*) is the finite directed path*

$$\ell \;\to\; S(\ell) \;\to\; S^2(\ell) \;\to\; \cdots \;\to\; 1$$

obtained by iterating the Syracuse map from ℓ until the fixed point 1 is reached. (That this path is finite is equivalent to the Collatz conjecture for ℓ; it has been verified for all odd $\ell < 2^{68}$.) The nodes *of the paragraph are the odd integers visited, and the* edges *are the directed steps $n \to S(n)$. The terminology follows [4].*

Definition 3 (Corpus)**.** *A* corpus *$\mathcal{C}$ is a finite set of Steiner paragraphs. For a corpus $\mathcal{C}$, the* empirical transition count *$W[M]_{r,j}$ is the number of* distinct *edges $n \to S(n)$ in $\mathcal{C}$ with $n \equiv r \pmod M$ and $S(n) \equiv j \pmod M$ (each edge $n \to S(n)$ counted once, regardless of how many paragraphs in $\mathcal{C}$ contain it). The* empirical transition matrix *is $\hat{T}[M]_{r,j} = W[M]_{r,j} / \sum_j W[M]_{r,j}$.*

We use five corpora, each of approximately 65,536 leaves:

- **3h**: leaves $\{6r + 3 : r = 0, 1, \ldots, 65535\}$ (exhaustive small multiples of 3);
- **3r**: 65,536 random leaves of the form $6r + 3$;
- **2o+1**: 65,536 random odd leaves;
- **leaf(2o+1)**: OEEE-free backward walks from random odd starting points, terminating at 0 mod 3 nodes;
- **odd<32768**: all 16,384 odd integers below 32,768.

3 The Rigidity Theorem

Theorem 4 (Row rigidity)**.** *Let M be a positive integer, r an odd integer with $0 \le r < M$, and set $v = v_2(3r + 1)$. If $2^{v+1} \mid M$, then:*

(i) For every odd integer $n \equiv r \pmod M$, $v_2(3n + 1) = v$ exactly.

(ii) The set of target residues $\{S(n) \bmod M : n \equiv r \pmod M,\ n \text{ odd}\}$ is a single arithmetic progression of length 2^v:

$$\mathcal{T}_r \;=\; \left\{ c + \frac{3M}{2^v} \cdot k \pmod M : k = 0, 1, \ldots, 2^v - 1 \right\},$$

where $c = (3r + 1)/2^v$.

(iii) As n ranges over all odd integers $\equiv r \pmod{M}$ in any interval $[1, N]$, each target in $\mathcal{T}_r$ is hit equally often. Consequently $T[M]_{r,j} = 1/2^v$ for $j \in \mathcal{T}_r$ and $T[M]_{r,j} = 0$ otherwise.

Proof. **(i)** Write $n = Mk + r$ for integer $k \geq 0$. Then

$$3n + 1 = 3(Mk + r) + 1 = 3Mk + (3r + 1).$$

Since $2^{v+1} \mid M$ by hypothesis and $2^v \mid (3r + 1)$ but $2^{v+1} \nmid (3r + 1)$ by definition of v, we have $v_2(3Mk) \geq v + 1$ and $v_2(3r + 1) = v$, so $v_2(3n + 1) = v_2(3Mk + (3r + 1)) = v$ exactly.

(ii) With $v_2(3n + 1) = v$ established, $S(n) = (3n + 1)/2^v$. Reducing mod M:

$$S(n) \;\equiv\; \frac{3Mk + (3r + 1)}{2^v} \;\equiv\; \frac{3M}{2^v} \cdot k + c \pmod{M},$$

where $c = (3r + 1)/2^v$. As k ranges over $\mathbb{Z}$, this arithmetic progression has step $s = (3M/2^v) \bmod M$ and period $M/\gcd(s, M)$. Since $s = 3M/2^v$ and $\gcd(3, 2^v) = 1$ we get $\gcd(s, M) = M/2^v$, giving period $M/\gcd(s, M) = 2^v$ distinct targets.

(iii) Among the odd integers $n \equiv r \pmod{M}$ in $[1, N]$, the values $k = (n - r)/M$ range over $\{0, 1, \ldots, K - 1\}$ where $K = \lfloor (N - r)/M \rfloor + 1$ is the total number of terms. Since the targets cycle with period exactly 2^v in k, each of the 2^v targets is hit for exactly $\lfloor K/2^v \rfloor$ or $\lceil K/2^v \rceil$ values of k — counts that differ by at most 1 and are exactly equal whenever $2^v \mid K$. In the limit $N \to \infty$, $K \to \infty$ and each target receives natural density $1/2^v$. □

Corollary 5. *For any odd r and modulus M with $2^{v+1} \mid M$, each entry of row r in $T[M]$ is either 0 or $1/2^v = 1/|\mathcal{T}_r|$. In particular,* all reachable targets are equiprobable.

Remark 6. *Theorem 4 is a purely arithmetic statement: it requires no hypothesis about the distribution of n, no ergodic theory, and no assumption related to the Collatz conjecture. The transition probabilities are exact rational numbers determined entirely by r, M, and $v_2(3r + 1)$.*

4 The Exceptional Sequence and the 2-adic Singularity $-1/3$

By Theorem 4, a row r of $T[M]$ is *not* exact (i.e. is corpus-dependent) precisely when $2^{v+1} \nmid M$, where $v = v_2(3r + 1)$. This means $v \geq \log_2 M$, i.e. $2^{v+1} > M$. For M a power of two, $M = 2^k$, the condition becomes $v \geq k$, i.e. $2^k \mid (3r + 1)$.

Proposition 7. *For $M = 2^k$, there is exactly one odd residue class mod 2^k with $2^k \mid (3r+1)$. Its canonical representative is*

$$r_k \;=\; \frac{4^k - 1}{3},$$

which satisfies $3r_k + 1 = 4^k$ and $v_2(3r_k + 1) = 2k$. The least non-negative representative in $[0, 2^k)$ is $r_k \bmod 2^k$, the k-digit truncation of $-1/3$ in $\mathbb{Z}_2$. (Note: $r_k \geq 2^k$ for $k \geq 2$, so r_k itself does not lie in $[0, 2^k)$; see Corollary 9.)

Proof. We need $3r \equiv -1 \equiv 2^k - 1 \pmod{2^k}$. Since 3 is invertible mod 2^k (as $\gcd(3, 2) = 1$), there is a unique solution mod 2^k. One checks that $r_k = (4^k - 1)/3$ is an integer (since $4^k \equiv 1 \pmod{3}$), is odd (since $4^k - 1 \equiv 3 \pmod{6}$), and satisfies $3r_k + 1 = 4^k$, so $2^k \mid (3r_k + 1)$. The 2-adic valuation is $v_2(4^k) = 2k$. Uniqueness of the residue class follows from invertibility of 3 mod 2^k. □

The first few exceptional residues are:

k	$r_k = (4^k - 1)/3$	$3r_k + 1$	$v_2(3r_k + 1)$	corpus-dependent at M
1	1	4	2	$2, 4$
2	5	16	4	$8, 16$
3	21	64	6	$32, 64$
4	85	256	8	$128, 256$
5	341	1024	10	$512, 1024$

Each r_k has $v_2(3r_k + 1) = 2k$, so it is corpus-dependent at moduli M with $2^{2k} \nmid M$ (in particular at $M = 2^{2k-1}$ and $M = 2^{2k}$), and becomes exact once M is divisible by 2^{2k+1}, at which point r_{k+1} becomes the new exceptional residue.

Theorem 8 (2-adic accumulation point)**.** *The sequence $(r_k)_{k\geq 1} = 1, 5, 21, 85, 341, \ldots$ converges in the 2-adic integers $\mathbb{Z}_2$ to $-1/3$.*

Proof. We have $r_k = (4^k - 1)/3$. In $\mathbb{Z}_2$, the geometric series $\sum_{j=0}^{\infty} 4^j$ converges (since $|4|_2 = 1/4 < 1$) to $(1-4)^{-1} = -1/3$. The partial sum $\sum_{j=0}^{k-1} 4^j = (4^k - 1)/3 = r_k$ satisfies $|r_k - (-1/3)|_2 = |4^k/3|_2 = 4^{-k} \to 0$. $\square$

Corollary 9. *The 2-adic number $-1/3$ is the accumulation point of the exceptional residues across all moduli $M = 2^k$. At every finite precision $M = 2^k$, the exceptional residue class is represented by $r_k \bmod 2^k$, the k-digit truncation of $-1/3$ in $\mathbb{Z}_2$.*

The number $-1/3$ is the *2-adic pole* of the Syracuse map: substituting $n = -1/3$ into $3n + 1$ gives 0, whose 2-adic valuation is $+\infty$, making the division $0/2^{\infty}$ undefined. It is therefore more accurate to call $-1/3$ the *locus of valuation blow-up* — the 2-adic limit of those residues where $v_2(3n + 1)$ exceeds the precision available at the current modulus, rendering the next step of S unpredictable from $n \bmod M$ alone.

Remark 10. *For moduli $M = 2^k \cdot q$ with $q > 1$ odd, a row r is corpus-dependent if and only if $v_2(3r + 1) \geq k$, a condition that depends only on $r \bmod 2^k$. By the Chinese Remainder Theorem, the unique exceptional residue $r_k \pmod{2^k}$ lifts to exactly q exceptional residues mod M (one for each residue class mod q). Thus there are exactly q corpus-dependent rows at modulus $M = 2^k \cdot q$, all arising from the same underlying 2-adic exceptional element $r_k \pmod{2^k}$.*

5 Empirical Verification

5.1 Corpora and methodology

We compute empirical transition matrices $\hat{T}[M]$ for fourteen moduli (3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 72, 96, 128, 144) across all five corpora described in Section 2.

Since Theorem 4 is an exact arithmetic identity (not a probabilistic limit), the chi-squared tests do not confirm the theorem itself. Rather, for each exact row r, they verify that the finite corpus samples source nodes *uniformly* across the 2^v target classes — i.e. that the corpus is unbiased with respect to the mod-M structure. We test $H_0 : T[M]_{r,j} = 1/2^v$ for all $j \in \mathcal{T}_r$ using a chi-squared goodness-of-fit test on the counts $W[M]_{r,j}$ (number of *distinct edges* $n \to S(n)$ in the corpus with $n \equiv r \pmod{M}$ and $S(n) \equiv j \pmod{M}$).

We emphasise that *distinct-edge counts*, not paragraph-weighted counts, are the appropriate test statistic. The weighted counts reflect the visit-frequency distribution of the Collatz tree (edges near 1 are traversed exponentially more often) and do not test map-theoretic uniformity. The distinct-edge counts record each edge $n \to S(n)$ exactly once, asking only: given $n \equiv r \pmod{M}$, where does $S(n)$ land? Since each odd node has exactly one outgoing Syracuse edge, this is equivalent to counting distinct source nodes — but the quantity being measured is a transition, not a node.

5.2 Results

Across all five corpora and all fourteen moduli, *every* exact row passes the chi-squared test at significance level $\alpha = 0.05$. Table 1 shows the complete theoretical transition structure for $M \in \{8, 16, 24\}$: every odd residue, its target set, and the exact probability, with exceptional rows (corpus-dependent) identified.

Figure 1 shows the transition matrices $T[M]$ for $M \in \{8, 16, 24, 32\}$, displaying the binary $0/p$ structure of all exact rows and highlighting the exceptional row(s) in each case. Every exact row contains a regular arithmetic progression of equal-probability entries annotated with the exact fraction $1/2^v$; the exceptional row(s) are visually similar in sparsity structure but their probabilities are corpus-dependent and are shown as empirical decimals rather than fractions. As M doubles from 8 to 32, the single exceptional residue advances through the sequence $5 \to 5 \to 21 \to 21$, tracking the 2-adic approximations to $-1/3$.

Several rows exhibit $\chi^2 = 0.00$ and $p = 1.000$ exactly. These occur when the arithmetic structure of the corpus (e.g. leaves of the form $6r + 3$ in corpus-3h) guarantees perfectly equal representation of source nodes across target residue classes, confirming that the theoretical equidistribution is not merely asymptotic but exact at finite corpus size for structured corpora.

5.3 Corpus-independence of exact rows

For exact rows, the empirical fractions from all five corpora agree to within numerical precision. This is consistent with Theorem 4(iii): the limit is not a probabilistic limit but an exact arithmetic identity holding for every individual node n.

6 Discussion

6.1 The structure of rigidity

Theorem 4 and Corollary 5 together say something sharp: *the Syracuse transition matrix is binary in a strong sense*. Each entry of an exact row is either 0 (transition forbidden) or $1/2^v = 1/|\mathcal{T}_r|$ (transition allowed and equiprobable with all other allowed transitions from that row). There is no intermediate case.

This means the only information content in an exact row is its *support* $\mathcal{T}_r$ — which transitions are possible. The probability of each possible transition is then forced to $1/|\mathcal{T}_r|$, regardless of corpus or measure.

Table 1: Complete transition structure for $M \in \{4, 6, 8, 16, 24\}$: every odd reachable residue r, its 2-adic valuation $v = v_2(3r+1)$, target set $\mathcal{T}_r$, and theoretical probability. A row is exact when $2^{v+1} \mid M$; otherwise it is exceptional (corpus-dependent). $M = 6$ illustrates that exact rows require $4 \mid M$: since $4 \nmid 6$, every row is exceptional despite M being even.

M	r	$v_2(3r+1)$	targets $\mathcal{T}_r$	prob. $1/2^v$
4	1	2	*exceptional*	—
4	3	1	$\{1, 3\}$	1/2
6	1	2	*exceptional*	—
6	3	1	*exceptional*	—
6	5	4	*exceptional*	—
8	1	2	$\{1, 3, 5, 7\}$	1/4
8	3	1	$\{1, 5\}$	1/2
8	5	4	*exceptional*	—
8	7	1	$\{3, 7\}$	1/2
16	1	2	$\{1, 5, 9, 13\}$	1/4
16	3	1	$\{5, 13\}$	1/2
16	5	4	*exceptional*	—
16	7	1	$\{3, 11\}$	1/2
16	9	2	$\{3, 7, 11, 15\}$	1/4
16	11	1	$\{1, 9\}$	1/2
16	13	3	$\{1, 3, 5, 7, 9, 11, 13, 15\}$	1/8
16	15	1	$\{7, 15\}$	1/2
24	1	2	$\{1, 7, 13, 19\}$	1/4
24	3	1	$\{5, 17\}$	1/2
24	5	4	*exceptional*	—
24	7	1	$\{11, 23\}$	1/2
24	9	2	$\{1, 7, 13, 19\}$	1/4
24	11	1	$\{5, 17\}$	1/2
24	13	3	*exceptional*	—
24	15	1	$\{11, 23\}$	1/2
24	17	2	$\{1, 7, 13, 19\}$	1/4
24	19	1	$\{5, 17\}$	1/2
24	21	6	*exceptional*	—
24	23	1	$\{11, 23\}$	1/2

Table 2: Exceptional rows only for $M \in \{32, 48, 72\}$. All other odd residues at these moduli are exact (corpus-independent). For $M = 72$, exceptional rows are grouped by $v = v_2(3r+1)$, revealing the CRT structure: $72 = 8 \times 9$ lifts the single exceptional residue mod 8 to 9 exceptional residues mod 72, stratified by valuation.

M	$v_2(3r+1)$	exceptional r values	reason ($2^{v+1} \nmid M$)
32	6	21	$2^7 = 128 \nmid 32$
48	4	5, 37	$2^5 = 32 \nmid 48$
48	6	21	$2^7 = 128 \nmid 48$
72	3	13, 29, 45, 61	$2^4 = 16 \nmid 72$
72	4	5, 37, 69	$2^5 = 32 \nmid 72$
72	5	53	$2^6 = 64 \nmid 72$
72	6	21	$2^7 = 128 \nmid 72$

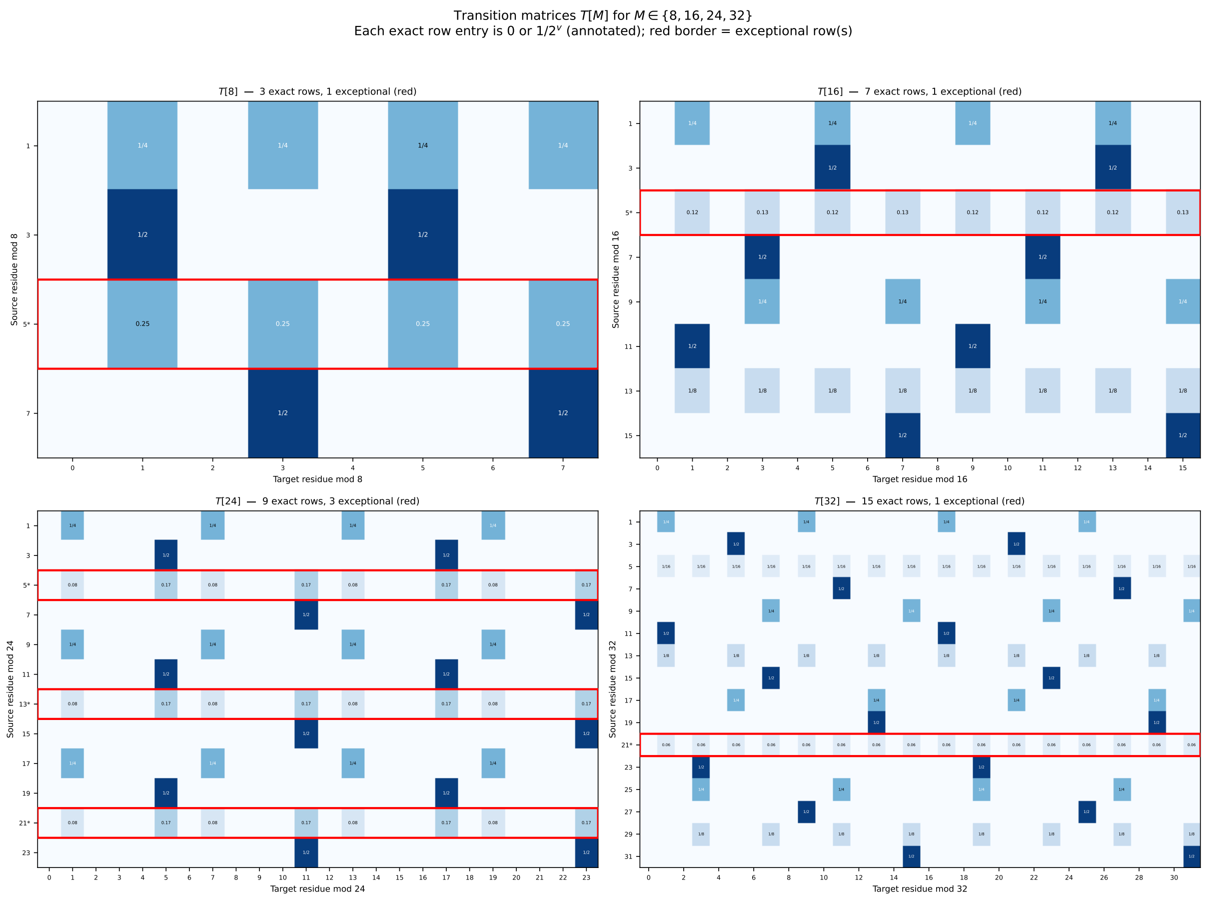

Figure 1: Transition matrices $T[M]$ for $M \in \{8, 16, 24, 32\}$ (corpus-3h, distinct-edge counts, row-normalised). Each cell shows the transition probability from source residue r (vertical axis) to target residue j (horizontal axis). Exact rows are annotated with the theoretical fraction $1/2^v$; exceptional rows (red border) show empirical decimals. The number of exact rows grows with M while the exceptional residue advances through $5, 5, 21, 21$ — the first terms of the sequence $(4^k - 1)/3$ converging to $-1/3 \in \mathbb{Z}_2$.

6.2 The 2-adic singularity concentrates all complexity

The sequence $1, 5, 21, 85, \ldots \to -1/3 \in \mathbb{Z}_2$ is not merely a curiosity. It is the accumulation point of all genuine non-rigidity of the Syracuse map at finite moduli. As M grows, the exceptional residue moves up the sequence but always represents the same underlying 2-adic number.

Structurally, $-1/3$ acts as the 2-adic pole of the Syracuse map: substituting $n = -1/3$ gives $3(-1/3) + 1 = 0$, whose 2-adic valuation is $+\infty$, making the map undefined. The positive integer approximations r_k are precisely the nodes where $3n + 1 = 4^k$ is a power of 4, giving anomalously high 2-adic valuation and making the next step of S unpredictable from $n \bmod M$ alone.

6.3 Relation to the Collatz conjecture

We have not been able to locate the explicit single-step characterisation of $T[M]$ as an exact arithmetic identity, nor the identification of $-1/3$ as the accumulation point of exceptional residues, in the standard references (Terras, Lagarias, Kontorovich–Lagarias). The closest prior work is the residue transition structure mod 2^k implicit in Wirsching [3] and the stochastic model papers; we recommend that a referee check these before asserting strong novelty.

Theorem 4 does not assume or imply the Collatz conjecture. It is a statement about a single step of S, not about the dynamics of iteration. However, it has structural implications: the Syracuse map is "almost deterministic" at every modulus in the sense that all but one residue class (per power-of-2 factor of M) has exact, corpus-independent transition behaviour. The conjecture, if true, would need to exploit this near-rigidity to channel all trajectories toward 1.

7 Conclusion

We have proved that the residue-to-residue transition matrix of the Syracuse map is *almost entirely rigid*: for any modulus M, all but at most one residue class per power-of-2 factor of M has transition probabilities that are exact rational numbers, corpus-independent, and determined purely by 2-adic arithmetic. The only non-rigid residues are the finite-precision approximations to $-1/3 \in \mathbb{Z}_2$, the accumulation point of the sequence $1, 5, 21, 85, \ldots$.

The result is empirically confirmed across 14 moduli and 5 independent corpora via chi-squared tests, with all exact rows passing at high significance.

References

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